Aalto University
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Spin Correlation Functions of the Planar Ising Model Using Discrete Complex Analysis and Orthogonal Polynomials

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Supervisor: Associate Professor Kalle Kytölä
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ABSTRACT OF
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The Ising model is a classical model in statistical physics, used to model ferromagnetic behavior of materials. This thesis focuses on the analysis of the diagonal spin-correlation functions in the 2D planar Ising model. The main results are the asymptotics of the spin correlation functions in critical and subcritical temperatures.

We approach the problem by defining a function on a lattice on the complex plane such that it has connections to the spin correlation functions. We use discrete complex analysis to show that this function satisfies a discrete analogue of Laplace's equation. Moreover, the discrete Fourier transform of the function allows us to express the problem of finding the spin correlations as a problem of finding polynomials that satisfy certain orthogonality conditions. In order to solve this problem, we discuss the theory of orthogonal polynomials on the unit circle, and show that the spin correlations can be expressed using the leading coefficients of orthogonal polynomials. In order to determine the asymptotic behavior of these coefficients, we prove the second Szegő theorem, which involves expressing the problem of finding orthogonal polynomials as a Riemann–Hilbert boundary value problem.

Keywords:	Ising model, orthogonal polynomials, spin correlation, mathematical physics, complex analysis
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Ising-malli on klassinen tilastollisen fysiikan malli, jota käytetään ferromagneettisten materiaalien mallintamiseen. Tässä työssä analysoidaan kaksiulotteisen neliöhilan Ising-mallin diagonaalisia spin-korrelaatioita. Työn päätulokset käsittelevät spin-korrelaatiofuntioiden asymptotiikkaa kriittisessä ja alikriittisessä lämpötilassa.

Ongelmaa lähestytään siten, että määritellään funktio hilalla kompleksitasolla siten, että sillä on yhteys spin-korrelaatiofunktioihin. Diskreetin kompleksianalyysin avulla näytetään, että tämä funktio toteuttaa Laplacen yhtälöä vastaavan diskreetin ehdon. Tämän funktion diskreetin Fourier-muunnoksen avulla spin-korrelaatiofunktioiden määrittäminen voidaan ilmaista siten, että etsitään polynomeja, jotka täyttävät tietyt ortogonaalisuusehdot. Tämän ongelman ratkaisemiseksi käsitellään yksikköympyrän ortogonaalisten polynomien teoriaa ja näytetään, että spin-korrelaatiofunktio voidaan ilmaista ortogonaalisten polynomien johtavien kertoimien avulla. Näiden kertoimien asymptoottisen käyttäytymisen määrittämistä varten todistetaan Szegő̋n toinen lause ilmaisemalla ortogonaalisten polynomien ongelma Riemann–Hilbert reuna-arvo-ongelmana.

Asiasanat:	Ising-malli, ortogonaalit polynomit, matemaattinen fysiikka, spin-korrelaatio, kompleksianalyysi
Kieli:	Englanti

Contents

Chapter 1

Introduction

The Ising model is a famous model in statistical physics. It is a simple approach to model the behavior of ferromagnetic materials. The Ising model was invented by the German physicist Wilhelm Lenz in 1920 [14]. He gave it as a problem to his student Ernst Ising, who solved the one dimensional model in his doctoral thesis [12] published in 1925.

The idea of the model is to divide solid matter into microscopic particles, and each particle is assigned a spin valued $+1$ or -1. The particles are assumed to have interactions with their neighboring particles. Each spin configuration of the entire system is assigned a probability using a probability measure that assigns higher probabilities to spin configurations that are energetically favorable, i.e. configurations where neighboring spins tend to align with each other. The temperature of the system is an important parameter that affects the probabilities.

The Ising model on a two dimensional planar graph is an interesting model to analyze because it is one of the simplest stochastic models that exhibits a phase transition. Having a phase transition means that the the qualitative behavior of the system changes sharply when a parameter is varied near a critical parameter value. An example of a phase transition in a physical system is how ferromagnetic materials behave when heat is applied to them. In a low temperature, ferromagnetic materials are in a ferromagnetic phase, in which spontaneous magnetization occurs in the material. When the temperature is raised above the Curie temperature of the material, the spontaneous magnetization vanishes. Above the critical temperature, the system is in paramagnetic phase.

The planar Ising model exhibits this kind of behavior. It is a simple way to model a uniaxial ferromagnet in different temperatures. The main results in this thesis focus on analyzing how the correlations between spin values behave over large distances. At subcritical temperatures, the spins of the

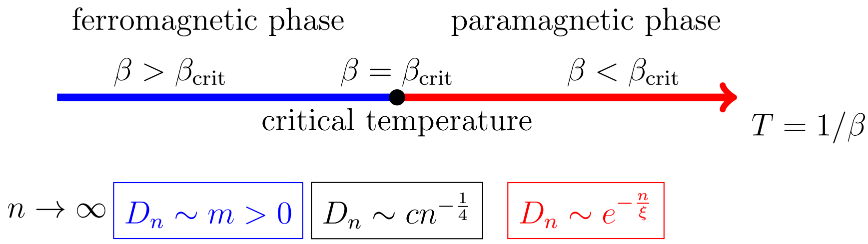

Figure 1.1: The spin correlations in a planar Ising model in different temperatures $T = \beta^{-1}$. The variable n is the distance between points and D_n is the spin correlation function while c and ξ are constants.

system tend to align with each other forming large chunks of particles that have the same spin. The spin values have positive correlation over arbitrarily large distances, as will be shown formally later. This is the ferromagnetic phase. At the critical temperature, this kind of behavior ceases and the correlations between spin values approaches zero at a polynomial rate of decay. This will be shown as a theorem later.

In a supercritical temperature, the spin correlations approaches zero at an exponential rate. This is the paramagnetic phase.

The phases and the behavior of the Ising model with different values of the temperature parameter are illustrated in Figure 1.1.

The spin correlations of the Ising model have been a subject of study for several decades. The results discussed above were known by the 60's. The researchers used techniques based on Toeplitz matrices in order to compute the spin correlation functions. The historical developments concerning this are discussed in [6]. The analysis of the Toeplitz matrices led to problems pertaining to othogonal polynomials on the unit circle.

Orthogonal polynomials on the unit circle are polynomials that, when integrated along the unit circle with respect to some weight function, are orthogonal to all the monomials of degrees lesser than the polynomial itself. The asymptotic behavior when the degree of the polynomial increases was a subject of special interest. Some important results are Szegő's limit theorems by Szegő [18][6].

Although the results derived have been known since the 60's [6], there are more recent approaches to analyzing the spin correlations of the Ising model. Dubédat has derived results using products of Ising correlators and free field correlators [7]. Chelkak, Hongler, and Izyurov have used discrete holomorphic spinor variables [4]. This discrete complex analytic approach allows proving conformal invariance for the Ising model and has connections to conformal field theory. Smirnov was awarded a Fields medal on his work on discrete holomorphic observables in 2010, which signifies how relevant the topic is in today's mathematical research. This thesis is based on the approach in [3] by Chelkak to prove results concerning diagonal spin correlations of the two dimensional Ising model.

First, in Chapter 2, we define the Ising model on finite and inifinite square lattices. We also prove combinatorial results related to the Ising model that we need later.

In Chapter 3, we define square lattices on the complex plane and define Ising models on those lattices. Then, we define a function on the lattice points. We will show that this function has discrete holomorphic properties (Theorem 3.1). Its discrete Fourier transform has properties that connect the spin correlation functions to certain orthogonal polynomials on the unit circle (Theorem 3.2).

In Chapter 4, we prove that at the critical temperature, a certain function satisfies the conditions for the Fourier transform in the previous chapter. Then, we use this function to prove that the spin correlation function is

$$D_n = \left(\frac{2}{\pi}\right)^n \prod_{k=1}^{n-1} \left(1 - \frac{1}{4k^2}\right)^{k-n} \sim 2^{1/3} e^{3\zeta'(-1)} (2n)^{-1/4}.$$

at the critical temperature (Theorem 4.1).

To analyze the correlation functions when the temperature is not critical, we need more theory about the orthogonal polynomials on the unit circle. In chapter 5, we define the orthogonal polynomials on the unit circle and prove some basic results that are useful later.

Chapter 6 focuses on analyzing the asymptotic behavior of leading coefficients of orthogonal polynomials when the degree is large. This is done using tools from complex analysis with functions that satisfy certain Riemann–Hilbert boundary value problems. We also prove the second Szegő Theorem (Theorem 6.1).

In Chapter 7, we use the results concerning orthogonal polynomials to prove that the spin correlation function approaches a strictly positive value at a subcritical temperature (Theorem 7.1).

Chapter 2

The Ising Model

This chapter first introduces and provides definitions of the planar Ising model on a finite square lattice with different boundary conditions. The high temperature expansion and low temperature expansion representations of the Ising model will be derived. Finally, the Ising model will be defined on the infinite square lattice.

2.1 Ising Model with Free Boundary Conditions

Let us define some graphs first. We define a graph that is a finite subset of the square lattice $\mathbb{Z}^2$. We also define its dual dual graph, which will be useful later.

Definition 2.1. *Let the finite graph $G \subset \mathbb{Z}^2$ consist of entire square faces. The edge set $E(G)$ connects the nearest neighbors in $\mathbb{Z}^{\nvDash}$. Let us define the dual graph G^* as the set of the square faces of G, the corresponding edge set $E(G^*)$ connecting the adjacent faces. Let ∂G^* be the set of square faces in $\mathbb{Z}^2$ adjacent to those in G^*. We denote $\overline{G}^* = G^* \cup \partial G^*$, and $E(\overline{G^*})$ connects the adjacent faces in $\overline{G}^*$.*

Let us consider the Ising model on G. We assign a spin $\sigma_x \in \{-1, +1\}$ for each node $x \in G$. All the spins on G form a configuration denoted by $\sigma = (\sigma_x)_{x \in G}$. The state space, i.e. the set of all possible configurations, is $\Omega = \{-1, +1\}^G$, and its size is $|\Omega| = 2^{|G|}$. An example configuration is illustrated in Figure 2.1.

Next, we define a probability measure on Ω. The probability of a configuration $\sigma \in \Omega$ is defined as

$$\mathbb{P}_G^{\text{free}}[\sigma] = \frac{1}{Z} e^{\beta \sum_{\{x,y\} \in E(G)} \sigma_x \sigma_y}, \tag{2.1}$$

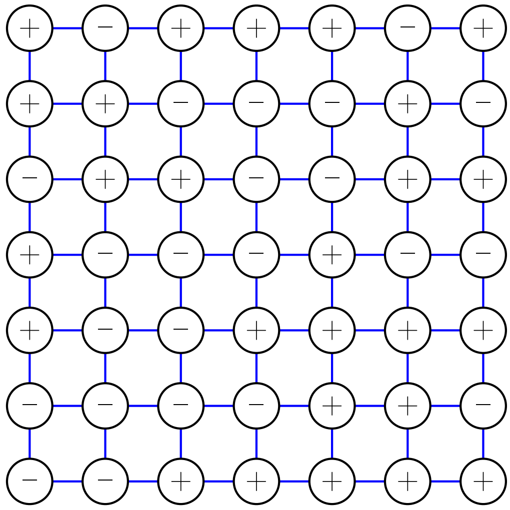

Figure 2.1: An illustration of a spin configuration on a finite subset of the square lattice

where the parameter $\beta > 0$ is called the inverse temperature, and Z is a normalization constant defined as

$$Z = \sum_{\sigma \in \Omega} e^{\beta \sum_{\{x,y\} \in E(G)} \sigma_x \sigma_y}.$$

The normalization constant is called the partition function. This probability measure defines the Ising model with free boundary conditions.

Let us consider the product $\sigma_x \sigma_y$, which appears inside the summation in the definition of the probability measure. The spins are valued ± 1, so the product takes the values

$$\sigma_x \sigma_y = \begin{cases} +1 & \text{if } \sigma_x = \sigma_y \\ -1 & \text{if } \sigma_x \neq \sigma_y. \end{cases}$$

Thus having the spins of the nearest neighbours align with each other leads to high probabilities compared to having them with the opposite signs.

The probability measure is of the form $e^{-E/T}$, where $T = \beta^{-1}$ is the temperature and E is the energy of the system. Thus this is an example of a Gibbs measure.

2.2 Ising Model with +-boundary Conditions

We define the Ising model on the dual graph G^* in an inverse temperature β^*. The configuration σ is defined on the graph G^* this time. The probability measure of the Ising model with +-boundary conditions is defined as

$$\mathbb{P}^{+}_{G^*}[\sigma] = \frac{1}{Z_+} e^{\beta^* \sum_{\{x,y\} \in E(\overline{G}^*)} \sigma_x \sigma_y},$$

where the partition function is

$$Z^{+} = \sum_{\sigma \in \Omega} e^{\beta^* \sum_{\{x,y\} \in E(\overline{G}^*)} \sigma_x \sigma_y},$$

and the boundary values are defined as $\sigma_x = +1$ for all $x \in \partial G^*$. The definition is mostly similar to the free boundary value case discussed above, with the main difference that now we are including edges from G^* to ∂G^* in the summations and assuming +-spins on the boundary.

2.3 Low Temperature Expansion

An even graph is a graph in which each node has an even degree, i.e. there are an even number of edges connected to each node (i.e. 0, 2 or 4 nodes in the case of a square lattice). We denote the collection of even subgraphs of the graph G by $\mathcal{E}(G)$ or just $\mathcal{E}$.

We will derive an alternate expression for the configuration probabilities of the Ising model with +-boundary conditions using even subgrahps on the dual graph. The representation is called the low temperature expansion, because it expresses the configuration probability as a power series in which the expansion parameter is small when the temperature low. This time we will consider the Ising model on the dual graph G^* at an inverse temperature β^*.

Proposition 2.1. *Let us consider an Ising model with +-boundary conditions on G^*. Let the inverse temperature be $\beta^* > 0$. The configuration probability can be expressed as*

$$\mathbb{P}^{+}_{G^*}[\sigma] = \frac{1}{\mathcal{Z}^+} \alpha^{|P|},$$

where the expansion parameter is $\alpha = e^{-2\beta^*}$, the graph $P = P(\sigma)$ is the even subgraph of G separating the domains of different spins on the dual G^*, and the normalization constant is

$$\mathcal{Z}^+ = \sum_{P \in \mathcal{E}(G)} \alpha^{|P|}.$$

Proof. A subgraph P has an edge between each two differing spins on $\overline{G}^*$. Figure 2.2 illustrates an example configuration on G^* and the corresponding domain walls on G. Let us consider the sum in (2.1). The sum will increase by one for each edge $\{x,y\} \in E(\overline{G}^*)$ satisfying $\sigma_x = \sigma_y$. If the spins differ, the sum decreases by one. There is an edge in P for each node pair of the latter case. For pairs according to the former case, there is no element in P. Thus, we have

$$\begin{aligned} e^{\beta^* \sum_{\{x,y\}\in E(\overline{G}^*)} \sigma_x\sigma_y} &= e^{\beta^*\left(|E(\overline{G}^*)\setminus P| - |P|\right)} = e^{\beta^*\left(|E(\overline{G}^*)|\right) - 2|P|} \\ &= e^{\beta^*|E(\overline{G}^*)|} e^{-2\beta^*|P|}. \end{aligned}$$

Similarly, the partition function can be expressed as

$$\mathcal{Z}^+ = \sum_\sigma e^{\beta^* \sum_{\{x,y\}\in E(\overline{G}^*)} \sigma_x\sigma_y} = e^{\beta^*|E(\overline{G}^*)|} \sum_{P\in\mathcal{E}} e^{-2\beta^*|P|}.$$

Substituting the above computations to the definition of $\mathbb{P}^+_{G^*}$ yields

$$\mathbb{P}^+_{G^*}[\sigma] = \frac{e^{\beta^*|E(\overline{G}^*)|} e^{-2\beta^*|P|}}{e^{\beta^*|E(\overline{G}^*)|}\sum_{P\in\mathcal{E}} e^{-2\beta^*|P|}} = \frac{e^{-2\beta^*|P|}}{\sum_{P\in\mathcal{E}} e^{-2\beta^*|P|}} = \frac{1}{\mathcal{Z}^+}\alpha^{|P|}.$$

□

2.4 High Temperature Expansion

Let us consider the Ising model with free boundary conditions on a graph G, which we still take to be a subgraph of $\mathbb{Z}^2$ as we defined earlier. We will derive an expression for the spin-spin expectations. It will be a power series in a parameter which becomes small in high temperatures, which is the reason it is called the high temperature expansion.

The notation $\mathcal{E}(v_1, \cdots, v_k)$ denotes the collection of subgraphs of G such that the nodes $v_1, \cdots, v_k$ have an odd degree and all the other nodes have an even degree. If a node is listed twice (or an even number of times), it has an even degree in the subgraph. If a node is listed an odd number of times, then it has an odd degree in the subgraph. Figure 2.3 represents an example of such a subgraph with parity broken at specific nodes.

Proposition 2.2. *For an Ising model on G with free boundary conditions, for all $v_1, \cdots, v_k \in G$, having an inverse temperature β, we have*

$$\mathbb{E}\left[\prod_{j=1}^k \sigma_{v_j}\right] = \frac{1}{\mathcal{Z}} \sum_{P\in\mathcal{E}(v_1,\cdots,v_k)} \alpha^{|P|},$$

where the expansion parameter is $\alpha = \tanh\beta$ and the normalization constant is $\mathcal{Z} = \sum_{P\in\mathcal{E}} \alpha^{|P|}$.

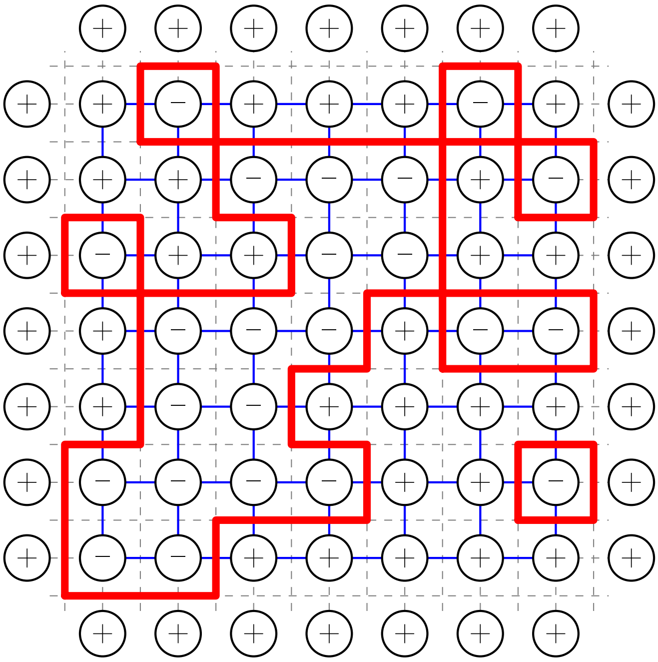

Figure 2.2: An example configuration with $+$-boundary conditions. The illustration has an even graph P (red) on the primal graph, separating the domains of differing spins on the dual G^* from each other.

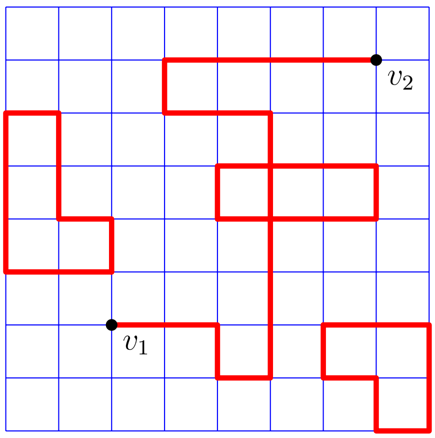

Figure 2.3: An example of a subgraph $P \in \mathcal{E}(v_1, v_2)$. The parity is broken at the nodes v_1 and v_2.

Proof. Using the configuration probability of the Ising model with free boundary conditions, we write the expectation as

$$\begin{aligned}
\mathbb{E}\left[\prod_{j=1}^{k}\sigma_{v_j}\right] &= \sum_{\sigma}\mathbb{P}_{\text{free}}[\sigma]\prod_{j=1}^{k}\sigma_{v_j} \\
&= \sum_{\sigma}\frac{1}{Z}e^{\beta\sum_{\{x,y\}\in E(G)}\sigma_x\sigma_y}\prod_{j=1}^{k}\sigma_{v_j} \\
&= \sum_{\sigma}\frac{1}{Z}\prod_{\{x,y\}\in E(G)}e^{\beta\sigma_x\sigma_y}\prod_{j=1}^{k}\sigma_{v_j}
\end{aligned}$$

Let us expand the expression for the partition function Z, and we have

$$\mathbb{E}\left[\prod_{j=1}^{k}\sigma_{v_j}\right] = \frac{\sum_{\sigma}\prod_{\{x,y\}\in E(G)}e^{\beta\sigma_x\sigma_y}\prod_{j=1}^{k}\sigma_{v_j}}{\sum_{\sigma}\prod_{\{x,y\}\in E(G)}e^{\beta\sigma_x\sigma_y}}. \tag{2.2}$$

It is easy to check that for $s\in\{-1,1\}$, the identities

$$e^{s\beta} = \cosh\beta + s\sinh\beta = \cosh\beta\,(1+s\tanh\beta)$$

hold. We have $\sigma_x\sigma_y\in\{-1,1\}$, so using the identity above, we write the numerator of (2.2) as

$$\begin{aligned}
&\sum_{\sigma}\prod_{\{x,y\}\in E(G)}\cosh\beta\,(1+\sigma_x\sigma_y\tanh\beta)\prod_{j=1}^{k}\sigma_{v_j} \\
=&(\cosh\beta)^{|E(G)|}\sum_{\sigma}\prod_{\{x,y\}\in E(G)}(1+\sigma_x\sigma_y\tanh\beta)\prod_{j=1}^{k}\sigma_{v_j}.
\end{aligned}$$

We use the binomial expansion on the product

$$\begin{aligned}
\prod_{\{x,y\}\in E(G)}(1+\sigma_x\sigma_y\tanh\beta) &= \sum_{A\subset E(G)}\prod_{\{x,y\}\in A}\sigma_x\sigma_y\tanh\beta \\
&= \sum_{A\subset E(G)}(\tanh\beta)^{|A|}\prod_{\{x,y\}\in A}\sigma_x\sigma_y,
\end{aligned}$$

and write the numerator of (2.2) as

$$(\cosh\beta)^{|E(G)|}\sum_{\sigma}\prod_{j=1}^{k}\sigma_{v_j}\sum_{A\subset E(G)}(\tanh\beta)^{|A|}\prod_{\{x,y\}\in A}\sigma_x\sigma_y.$$

Changing the order of summation and substituting $\tanh\beta = \alpha$ leads to

$$
\begin{aligned}
&(\cosh\beta)^{|E(G)|} \sum_{A \subset E(G)} \alpha^{|A|} \sum_{\sigma} \left(\prod_{\{x,y\} \in A} \sigma_x \sigma_y \right) \left(\prod_{j=1}^{k} \sigma_{v_j} \right) \\
&= (\cosh\beta)^{|E(G)|} \sum_{A \subset E(G)} \alpha^{|A|} \sum_{\sigma} \prod_{x \in G} \sigma_x^{\deg_A(x) + \mathbb{1}_{\{v_1, \cdots, v_k\}}(x)},
\end{aligned} \tag{2.3}
$$

where the notation $\deg_A(x)$ means the degree of the node x on graph A, and $\mathbb{1}_B(x)$ is the indicator having the value 1 if x is in B, 0 otherwise.

Let us study how the last sum behaves for different subsets A. There are two distinct cases:

1. $A \in \mathcal{E}(v_1, v_2, \cdots, v_k)$
2. $A \notin \mathcal{E}(v_1, v_2, \cdots, v_k)$.

First, let us assume that we have $A \in \mathcal{E}(v_1, v_2, \cdots, v_k)$. Then $\deg_A(x)$ is even for all $x \notin \{v_j\}_{j=1}^k$ and odd for all $x \in \{v_j\}_{j=1}^k$. Each spin σ_x is multiplied by itself $\deg_A(x) + \mathbb{1}_{\{v_1, \cdots, v_k\}}(x)$ times, that is an even number for all x. Therefore, the product always has the value 1 and

$$
\sum_{\sigma} \prod_{x \in A} \sigma_x^{\deg_A(x) + \mathbb{1}_{\{v_1, \cdots, v_k\}}(x)} = |\Omega| = 2^{|G|}
$$

holds.

Next, let us consider the case with $A \notin \mathcal{E}(v_1, v_2, \cdots, v_k)$. This means that either there exists a node $x \notin \{v_j\}_{j=1}^k$ for which $\deg_A(x)$ is odd, or there exists a node $x \in \{v_j\}_{j=1}^k$ for which $\deg_A(x)$ is even. Hence, $\deg_A(x) + \mathbb{1}_{\{v_1, \cdots, v_k\}}(x)$ is odd implying that $\sigma_x^{\deg_A(x) + \mathbb{1}_{\{v_1, \cdots, v_k\}}(x)} = \sigma_x$. Let us consider an arbitrary spin configuration σ. There is exactly one configuration σ', which is the same as σ, except that it has a different spin at the point x. Therefore, when we sum over all the possible configurations in the second summation in (2.3), for each configuration there is another configuration that has the opposite sign for the product $\prod_{x \in A} \sigma_x^{\deg_A(x) + \mathbb{1}_{\{v_1, \cdots, v_k\}}(x)}$, which implies that the entire sum equals zero.

By combining the results from the two cases discussed above, we conclude that the numerator of (2.2) can be written as

$$
(\cosh\beta)^{|E(G)|} 2^{|G|} \sum_{P \in \mathcal{E}(v_1, \cdots, v_k)} \alpha^{|P|}.
$$

The denominator of (2.2) is simply the numerator with an empty list of points (v_j), so we have

$$\mathbb{E}\left[\prod_{j=1}^{k}\sigma_{v_j}\right] = \frac{(\cosh\beta)^{|E(G)|}2^{|G|}\sum_{P\in\mathcal{E}(v_1,\cdots,v_k)}\alpha^{|P|}}{(\cosh\beta)^{|E(G)|}2^{|G|}\sum_{P\in\mathcal{E}}\alpha^{|P|}}$$
$$= \frac{\sum_{P\in\mathcal{E}(v_1,\cdots,v_k)}\alpha^{|P|}}{\sum_{P\in\mathcal{E}}\alpha^{|P|}} = \frac{1}{\mathcal{Z}}\sum_{P\in\mathcal{E}(v_1,\cdots,v_k)}\alpha^{|P|}.$$

□

2.5 The Thermodynamic Limit of the Ising Model

To analyze the Ising model on a macroscopic scale, one often defines it on an infinite square lattice. The following results show that when the size of the system approaches infinity, the probability measures converge to thermodynamic limits.

Let $(G_\ell)_{\ell=1}^{\infty}$ be an increasing sequence of finite subgraphs of $\mathbb{Z}^2$, such that $G_\ell \subset G_{\ell+1}$ holds for all ℓ, and we have $\bigcup_{\ell=0}^{\infty} G_\ell = \mathbb{Z}^2$. Let us define a configuration $\sigma \in \{-1,+1\}^{\mathbb{Z}^2}$. It is well known that the probability measures for Ising models with free or positive boundary conditions have weak limits

$$\mathbb{P}^{\text{free}}_{G_\ell} \to \mathbb{P}^{\text{free}}_{\mathbb{Z}^2}$$

and

$$\mathbb{P}^{+}_{G_\ell} \to \mathbb{P}^{+}_{\mathbb{Z}^2},$$

as we let $\ell \to \infty$. The limits are unique regardless the choice of the increasing sequence. See [13] and [10, ch. 4] for further details.

The existence of the limits also implies that it is reasonable to study spin–spin-expectations in the infinite Ising model.

The critical inverse temperature β_{crit} is characterized by the property that the expansion parameters for the high temperature expansion and low temperature expansion are the same, i.e we have $e^{-2\beta} = \tanh\beta$. This is satisfied for $\beta = \beta_{\text{crit}} = \frac{1}{2}\log\left(1+\sqrt{2}\right)$ [2] [17].

Moreover, at the critical temperature $\beta = \beta_{\text{crit}}$, the limiting measures are the same for all boundary conditions. Especially, we have

$$\mathbb{P}^{\text{free}}_{\mathbb{Z}^2} = \mathbb{P}^{+}_{\mathbb{Z}^2} \tag{2.4}$$

at the critical temperature [21, Chapter 10].

Chapter 3

From the Ising Model to Orthogonal Polynomials

In this chapter, we will show how to connect the spin-spin expectations of the Ising model to the orthogonal polynomials. In order to do so, we will define a discrete complex function that we call Θ. Then we will show that this function has discrete holomorphicity properties. Finally, it will be established that using the discrete Fourier transform of this function, it is possible to reformulate the problem of finding the spin-spin expectations as a problem of finding orthogonal polynomials.

3.1 Auxiliary Function Θ

Let us define two square lattices on the complex plane $\mathbb{C}$. The first one is

$$G = \{k + \mathbb{i}s \mid k, s \in \mathbb{Z}, k + s \equiv 1 \pmod 2\},$$

which is a square lattice rotated by $\pi/4$. The second one is

$$G^* = \{k + \mathbb{i}s \mid k, s \in \mathbb{Z}, k + s \equiv 0 \pmod 2\},$$

which is the dual lattice of G consisting of its faces. The setup is illustrated in Figure 3.1.

Let us consider a node $p \in G$. We call the points $\{p+\frac{1}{2}, p-\frac{1}{2}, p+\frac{1}{2}\mathbb{i}, p-\frac{1}{2}\mathbb{i}\}$ the nearest corners of p.

Let $G_\ell = \{k + \mathbb{i}s \mid k, s \in \mathbb{Z}, |k| + |s| \le \ell\}$, i.e. the graphs G_ℓ are rotated square-shaped subsets of G. Let $G^*_\ell \subset G^*$ consists of the faces of G_ℓ. In later analysis we could use some other increasing sequence of subgraphs of G that satisfies certain symmetry conditions, but the choice would not affect the results.

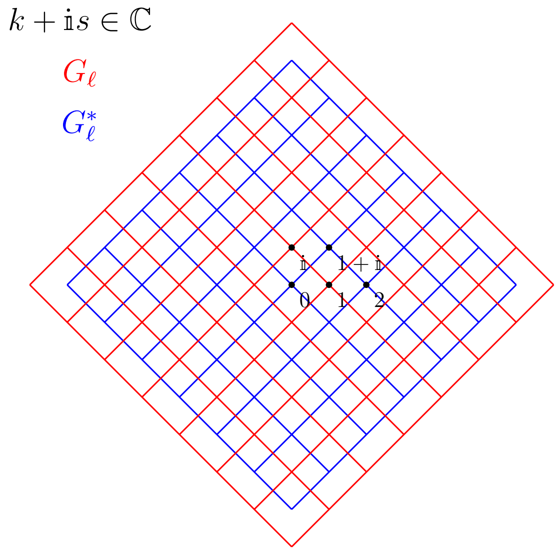

Figure 3.1: An illustration of a graph G_ℓ and its dual G_ℓ^* on the complex plane.

Definition 3.1. *Let c be a corner of G, which means that c is of the form $c = k + \mathbb{i}s + z$, where we have $k + \mathbb{i}s \in G$ and $z \in \{\pm\mathbb{i}/2, \pm 1/2\}$. Let β be the inverse temperature of the Ising model on graph G. Let β^* be the inverse temperature on the Ising model on the dual graph G^*. Let us choose β and β^* so that for some number α, the relation*

$$\tanh\beta = e^{-2\beta^*} = \alpha$$

is satisfied.

Let us define a function on the corners of G as

$$\Theta_n^{(\ell)}(c) = \frac{1}{\mathcal{Z}_\ell} \sum_{P \in \mathcal{E}_\ell(-1, k-1+\mathbb{i}s)} \alpha^{|P|} e^{-\frac{\mathbb{i}}{2}\mathcal{W}(\pi(P))} (-1)^{\mathrm{loops}_{-2,2n}(P \setminus \pi(P))} (-1)^{\mathrm{sheet}(\pi(P))},$$

where the terms are defined as explained below.

We define $\mathcal{E}_\ell(v_1, v_2) = \mathcal{E}(v_1, v_2) \cap \{P \mid P \subset G_\ell\}$. The graph P is an even subraph of G_ℓ, except at the points -1 and $k - 1 + \mathbb{i}s$. Thus there has to be a path in P connecting these two points. We denote this path between -1 and $k - 1 + \mathbb{i}s$ in P by $\pi(P)$. The path might not be unique if there are nodes of degree 4 in $\pi(P)$. In this case, we split the graph at the intersection point in any way such that the resulting path does not intersect itself.

The winding $\mathcal{W}(\pi(P))$ of the path between the points -1 and $k - 1 + \mathbb{i}s$ is the cumulative angle of turns along the path, when we define the path to start at angle 0 and to stop at angle pointing to the direction of z.

The term $\text{loops}_{-2,2n}(P \setminus \pi(P))$ *is the number of loops in* $P \setminus \pi(P)$ *enclosing exactly one of the points* $\{-2, 2n\}$.

Let us define branch cuts for the function. The branch cuts will the include all the edges $\{x, y\} \in E(G^*)$, *for which* $x = k$ *and* $y = k' - \mathbb{i}$ *for some* $k, k' \leq -2$ *or* $k, k' \geq 2n$. *The term* $\text{sheet}(\pi(P))$ *is the number of times the path* $\pi(P)$ *crosses the branch cuts.*

The number $|P|$ *is the total number of edges in the subgraph* P.

The normalization constant $\mathcal{Z}_\ell$ *is defined as*

$$\mathcal{Z}_\ell = \sum_{P \in \mathcal{E}(G_\ell)} \alpha^{|P|}.$$

Figure 3.2 demonstrates what a subgraph P looks like how to compute the corresponding terms in the definition of $\Theta_n^{(\ell)}$. The individual terms in the computation are not necessarily unique if we have the need to resolve intersections in the path $\pi(P)$. However, the product

$$e^{-\frac{\mathbb{i}}{2}\mathcal{W}(\pi(P))}(-1)^{\text{loops}_{-2,2n}(P \setminus \pi(P))}(-1)^{\text{sheet}(\pi(P))}$$

is well-defined regardless of how we choose to resolve the intersections.

We defined the function using finite subgraphs. However, we are interested in the properties of the infinite Ising model. The following lemma shows that $\Theta_n^{(\ell)}$ has subsequential limiting values at all the corners simultaneously, as ℓ increases.

Lemma 3.1. *There exists a subsequence* $(\ell_m)_{m \in \mathbb{N}}$ *such that for every corner* c, *the limit* $\lim_{m\to\infty} \Theta_n^{(\ell_m)}(c)$ *exists.*

Proof. First, we note that $\Theta_n^{(\ell)}$ is bounded, because by Proposition 2.2 (the high temperature expansion) we have

$$\left|\Theta_n^{(\ell)}(k + \mathbb{i}s - \tfrac{1}{2})\right| \leq \left| \frac{1}{\mathcal{Z}_\ell} \sum_{P \in \mathcal{E}_\ell(-1, k-1+\mathbb{i}s)} \alpha^{|P|} \right| = |\mathbb{E}[\sigma_{-1}\sigma_{2n-1}]| \leq 1,$$

as the expectation of a 0-1-valued random variable has to be 1 at most. Let $(p_j)_{j=0}^{\infty}$ be an enumeration of the countable set of all corners c of the square lattice.

Since $\Theta_n^{(\ell)}(p_0)$ is bounded, there exists a subsequence $\ell_m^{(0)}$ such that the sequence $\Theta_n^{(\ell_m^{(0)})}(p_0)$ converges as $m \to \infty$. Also, $\Theta_n^{(\ell)}(p_1)$ is bounded so $\ell_m^{(0)}$ has a subsequence $\ell_m^{(1)}$ such that $\Theta_n^{(\ell_m^{(1)})}(p_1)$ converges. Since $\ell_m^{(1)}$ is a subsequence of $\ell_m^{(0)}$, $\Theta_n^{(\ell_m^{(1)})}(p_0)$ converges, as well. Similarly, we construct sequence $\ell_m^{(j)}$ as

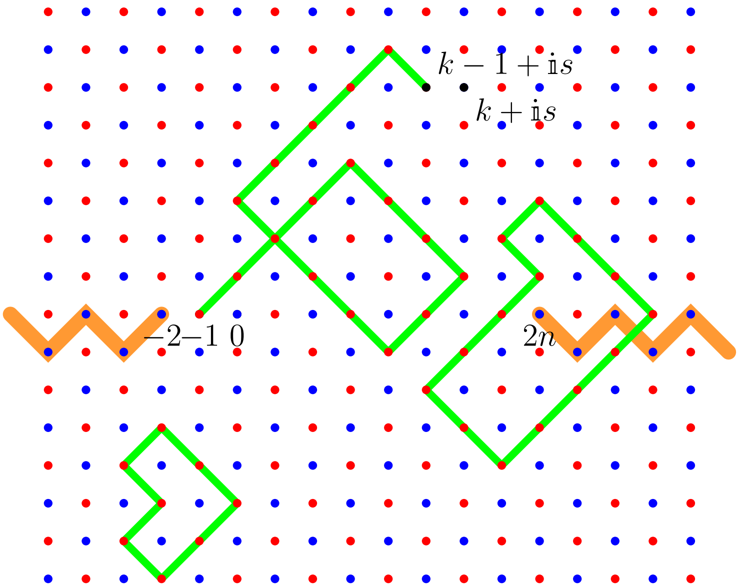

Figure 3.2: An example of a path P (green) from the definition of $\Theta_n^{(\ell)}$. The branch cuts are the orange lines. In this particular case, $\mathcal{W}(P) = 0$, $\text{loops}_{-2,2n}(P \setminus \pi(P)) = 1$, $\text{sheet}(\pi(P)) = 0$ and $|P| = 40$.

a subsequence of $\ell_m^{(j-1)}$ such that $\Theta_n^{(\ell_m^{(j)})}(p_j)$ converges. Note, that $\Theta_n^{(\ell_m^{(j)})}(p_i)$ converges for all $0 \leq i \leq j$. Consider the sequence $\ell_m^{(\text{diag})} = \ell_m^{(\ell)}$. Now $\Theta_n^{(\ell_m^{(\text{diag})})}(p_j)$ converges for all j. $\square$

We use the notation Θ_n to denote a subsequential limit of $\Theta_n^{(\ell)}$ in the topology of pointwise convergence, as $\ell \to \infty$. It will be proven later that the limit is unique, that is, it does not depend on the choice of the converging subsequence.

3.2 Some Properties of the Function Θ

In order to further study the properties of the function Θ_n, we define a discrete Laplace-like operator. Then we will study some essential properties of the function Θ_n.

Definition 3.2. *Let us consider a corner $z = k + \mathbb{i}s + z$, where we have $k, i \in \mathbb{Z}$ and $z \in \{\pm\mathbb{i}/2, \pm 1/2\}$. Let us define the operator Δ_θ, that operates* *on functions on the corners, such that we have*

$$\Delta_\theta f(z)$$
$$= f(z) - \frac{1}{4}\sin(2\theta)\Big(f(z+1+i) + f(z-1+i) + f(z+1-i) + f(z-1-i)\Big).$$

for all the corners with $s = 0, k < -2$ or $k > 2n$, and the constant $\theta \in \mathbb{R}$. Otherwise (that is in the proximity of the branch cuts) we define

$$\Delta_\theta f(z)$$
$$= f(z) - \frac{1}{4}\sin(2\theta)\Big(f(z+1+i) + f(z-1+i) - f(z+1-i) - f(z-1-i)\Big).$$

The following lemma assigns different different subspaces in $\mathbb{C}$ to corners, which will be useful later.

Lemma 3.2. *Let us consider a corner $c_z = k + \mathbb{i}s + \frac{1}{2}z$, where $z \in \{\pm 1, \pm \mathbb{i}\}$. The function Θ has values that belong to different subspaces of $\mathbb{C}$ according to the value of z. We have*

$$\Theta(c_1) \in \mathbb{R},$$
$$\Theta(c_{-1}) \in \mathbb{i}\mathbb{R},$$
$$\Theta(c_{\mathbb{i}}) \in e^{-\mathbb{i}\frac{\pi}{4}}\mathbb{R},$$
$$\Theta(c_{-\mathbb{i}}) \in e^{\mathbb{i}\frac{\pi}{4}}\mathbb{R}.$$

Proof. The winding $\mathcal{W}(\pi(P))$ as defined in Definition 3.1, is of the form $\arg(z) + 2\pi n$ for some $n \in \mathbb{Z}$ since the angle in which the path starts is predefined. Therefore $e^{-\frac{\mathbb{i}}{2}\mathcal{W}(\pi(P))}$ has values as described above. All the other parts in Definition 3.1 are real-valued, so the multiplication does not affect the argument of $\Theta(c_z)$. □

We are interested in the diagonal spin correlations. Let us define them for the Ising models on G and its dual G^*. The existence of the limits below is a consequence of the existence of the thermodynamical limits described in Section 2.5.

Definition 3.3. *We define the diagonal spin correlations on G^* as*

$$D_n^* = \mathbb{E}^*[\sigma_0\sigma_{2n}] = \lim_{\ell\to\infty} \mathbb{E}^+_{G^*_\ell}[\sigma_0\sigma_{2n}],$$

and on G as

$$D_n = \mathbb{E}[\sigma_{-1}\sigma_{2n-1}] = \lim_{\ell\to\infty} \mathbb{E}^{free}_{G_\ell}[\sigma_{-1}\sigma_{2n-1}],$$

where $(G_\ell)_{\ell=1}^\infty$ and $(G^*_\ell)_{\ell=0}^\infty$ are increasing sequences of subgraphs of $\mathbb{Z}^2$ as described in Section 2.5, and $\mathbb{E}^+_{G^*_\ell}$ and $\mathbb{E}^{free}_{G_\ell}$ are the expected values with respect to the probability measures $\mathbb{P}^+_{G^*_\ell}$ and $\mathbb{P}^{free}_{G_\ell}$, respectively.

The following theorem connects the function Θ_n with the diagonal spin correlations. It will play an important part later when establishing the connection between the orthogonal polynomials and the Ising model.

Theorem 3.1. *Let Θ_n be a limiting function as described in Lemma 3.1. It has the following properties:*

1. $\Theta_n(-\frac{1}{2}) = \mathbb{E}^*[\sigma_0\sigma_{2n}] = D_n^*$
2. $\Theta_n(2n-\frac{1}{2}) = \mathbb{E}[\sigma_{-1}\sigma_{2n-1}] = D_n$
3. $\Theta_n(k-\frac{1}{2}) = 0$ *for all* $k \leq -2$ *and for all* $k \geq 2n+2$
4. $\Theta_n(k+\mathbb{i}s-\frac{1}{2}) = \Theta_n(k-\mathbb{i}s-\frac{1}{2})$ *for all* k, s
5. $\Delta_\theta\Theta_n(k+\mathbb{i}s-\frac{1}{2}) = 0$ *for all* $k+\mathbb{i}s \notin \{0, 2n\}$
6. $\Delta_\theta\Theta_n(-\frac{1}{2}) = \frac{q^2}{1+q^2}D_{n+1}^*$, $\Delta_\theta\Theta_n(2n-\frac{1}{2}) = \frac{1}{1+q^2}D_{n+1}$,

where the parameter θ satisfies $e^{-2\beta^*} = \tan\frac{\theta}{2}$, and the parameter q is a real number such that $\sin(2\theta) = \frac{2}{q+q^{-1}}$.

Proof of part 1. According to the definition,

$$\Theta_n^{(\ell)}(-\tfrac{1}{2}) = \frac{1}{\mathcal{Z}_\ell} \sum_{P\in\mathcal{E}_\ell(-1,-1)} \alpha^{|P|} e^{-\frac{\mathbb{i}}{2}\mathcal{W}(\pi(P))}(-1)^{\mathrm{loops}_{-2,2n}(P\setminus\pi(P))}(-1)^{\mathrm{sheet}(\pi(P))}. \tag{3.1}$$

The path $\pi(P)$ only consists of the degenerate path $\{-1,-1\}$, or alternatively it forms a loop.

If $\pi(P)$ is the degenerate path of length zero, then the winding is zero and the path does not cross the branch cuts, so we have

$$e^{-\frac{\mathbb{i}}{2}\mathcal{W}(\pi(P))} = (-1)^{\mathrm{sheet}(\pi(P))} = 1. \tag{3.2}$$

Every loop in P enclosing the point -2 also encloses the point 0 because otherwise the loop in question would go through the point -1 thus becoming part of the path $\pi(P)$. This implies that we have

$$\mathrm{loops}_{-2,2n}(P\setminus\pi(P)) = \mathrm{loops}_{0,2n}(P\setminus\pi(P)) = \mathrm{loops}_{0,2n}(P). \tag{3.3}$$

Substituting (3.2) and (3.3) to the expression inside the sum in (3.1) yields

$$\alpha^{|P|}(-1)^{\mathrm{loops}_{0,2n}(P)}.$$

If $\pi(P)$ is a loop enclosing exactly one of the points 0 and -2, then we have $e^{-\frac{\mathrm{i}}{2}\mathcal{W}(\pi(P))} = -1$. Otherwise, we have $e^{-\frac{\mathrm{i}}{2}\mathcal{W}(\pi(P))} = 1$. The path crosses the branch cuts an odd number of times if it encloses the point -2, else the branch cuts are crossed an even number of times. Thus we have

$$e^{-\frac{\mathrm{i}}{2}\mathcal{W}(\pi(P))}(-1)^{\mathrm{sheet}(\pi(P))} = \begin{cases} -1 & \text{if } \pi(P) \text{ encloses the point 0 but not -2} \\ 1 & \text{otherwise.} \end{cases}$$

Every loop in $P \setminus \pi(P)$ enclosing the point -2 also encloses the point 0 as discussed above, so

$$e^{-\frac{\mathrm{i}}{2}\mathcal{W}(\pi(P))}(-1)^{\mathrm{loops}_{-2,2n}(P\setminus\pi(P))}(-1)^{\mathrm{sheet}(\pi(P))} = (-1)^{\mathrm{loops}_{0,2n}(P)}.$$

By combining this with the similar result for the case of the degenerate path, we conclude that

$$\Theta_n^{(\ell)}(-\tfrac{1}{2}) = \frac{1}{\mathcal{Z}_\ell} \sum_{P \in \mathcal{E}_\ell} \alpha^{|P|}(-1)^{\mathrm{loops}_{0,2n}(P)}$$

holds.

Next, we need to show that this converges to D_n^*. We write $D_n^* = \lim_{\ell\to\infty} \mathbb{E}^+_{G^*_\ell}[\sigma_0\sigma_{2n}]$, and observe that

$$\mathbb{E}^+_{G^*_\ell}[\sigma_0\sigma_{2n}] = \sum_\sigma \mathbb{P}^+_{G^*_\ell}[\sigma]\,\sigma_0\sigma_{2n} = \frac{1}{\mathcal{Z}_\ell}\sum_{P\in\mathcal{E}_\ell}\alpha^{|P|}\sigma_0(P)\sigma_{2n}(P)$$

using Proposition 2.1. Let us consider an individual term of the sum. It has one of the values $\pm\alpha^{|P|}$, depending on the sign of $\sigma_0\sigma_{2n}$. Let π' be an arbitrary path between the points 0 and $2n$. The path π' may cross a loop $L \subset P$ only an even number of times if L does not enclose exactly one of the points 0 and $2n$. Otherwise, L is crossed an odd number of times. The edges in P can be interpreted as domain walls separating the differing signs of spins from each other, as we discussed in Section 2.3 when defining the low temperature expansion of the Ising model. Therefore, if L crosses a domain wall an odd number of times, then we have $\sigma_0\sigma_{2n} = -1$, otherwise $\sigma_0\sigma_{2n} = 1$. Hence, we have

$$\sigma_0\sigma_{2n} = (-1)^{\mathrm{loops}_{0,2n}(P)}$$

and

$$D_n^* = \lim_{\ell\to\infty}\sum_{P\in\mathcal{E}_\ell}\alpha^{|P|}(-1)^{\mathrm{loops}_{0,2n}(P)} = \lim_{\ell\to\infty}\Theta_n^{(\ell)}(-\tfrac{1}{2}).$$

Proof of part 2. There has to be a crossing of a branch cut corresponding to each full $\pm 2\pi$ of winding, which implies that we have

$$e^{-\frac{\mathbb{i}}{2}\mathcal{W}(\pi(P))}(-1)^{\mathrm{sheet}(\pi(P))} = 1$$

for all P as the terms in the product cancel each other. Also, any loop enclosing just one of the points 0 and $2n$ has to cross the path thus becoming part of it. Hence, $(-1)^{\mathrm{loops}_{-2,2n}(P\setminus\pi(P))} = 1$. By comparing with Proposition 2.2, we see that the resulting expression is the spin correlation according to the high temperature expansion of the Ising model, yielding

$$\lim_{\ell\to\infty} \Theta_n^{(\ell)}(2n - \frac{1}{2}) = \lim_{\ell\to\infty} \frac{1}{\mathcal{Z}_\ell} \sum_{P\in\mathcal{E}_\ell(-1,2n-1)} \alpha^{|P|} = D_n.$$

Proof of part 3. Let us consider $k \leq -2$ or $k \geq 2n+2$ and $s = 0$. For $P \in \mathcal{E}_\ell(-1, k-1)$, reflecting across the k-axis results in the graph P'. The winding is a multiple of 2π, and winding after the reflection is negative of that. The reflected path P' has either one more or one less crossings of branch cuts compared to P. The terms corresponding to P and P' cancel each other out when computing the sum in the definition of $\Theta_n^{(\ell)}(k - \frac{1}{2})$.

Proof of part 4. Let us consider a graph P, which is one of the elements in the summation set in the definition of $\Theta_n^{(\ell)}(k + \mathbb{i}s - \frac{1}{2})$. Let us reflect P across the k-axis to obtain a graph P'. The graph P' contains a path from the point -1 to $k - \mathbb{i}s - 1$, and thus it is one of the graphs in the summation when computing $\Theta_n^{(\ell)}(k - \mathbb{i}s - \frac{1}{2})$. Clearly, we have

$$\mathrm{loops}_{-2,2n}\left(P' \setminus \pi(P')\right) = \mathrm{loops}_{-2,2n}\left(P \setminus \pi(P)\right)$$

and

$$\mathrm{sheet}\left(\pi(P')\right) = \mathrm{sheet}\left(\pi(P)\right).$$

The reflecting causes the winding to have the opposite sign, so it is

$$\mathcal{W}\left(\pi(P')\right) = -\mathcal{W}\left(\pi(P)\right).$$

Since the winding is a multiple of $\pm 2\pi$ in this case, we have

$$e^{-\frac{\mathbb{i}}{2}\mathcal{W}(\pi(P))} = e^{-\frac{\mathbb{i}}{2}\mathcal{W}(\pi(P'))}$$

for all P. Thus we have $\Theta_n^{(\ell)}(k + \mathbb{i}s - \frac{1}{2}) = \Theta_n^{(\ell)}(k - \mathbb{i}s - \frac{1}{2})$ for all n, so $\Theta_n(k + \mathbb{i}s - \frac{1}{2}) = \Theta_n(k - \mathbb{i}s - \frac{1}{2})$ holds.

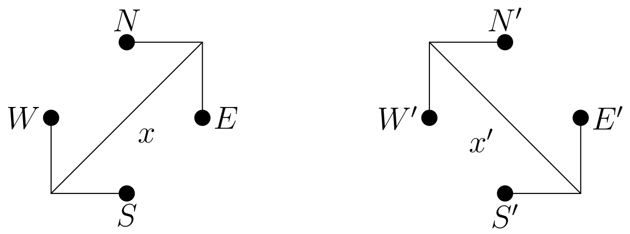

Figure 3.3: The two types of edges and the naming of the nearest corners. Here x is of the type / and x' of the type \.

Proof of part 5. Let us consider an edge $x \in E(G_\ell)$. There exist two types of edges as presented in Figure 3.3. Let us call these type / and type \. We shall name the four nearest corners north, south, east and west, or N, S, W and E, according to Figure 3.3. Each individual corner can have two different names depending on which edge we are observing, because a corner is a neighbor to two edges of different types.

Let us assume that x is of type /. Next, we will show that then relation

$$\Theta_n(N) - \Theta_n(S) = e^{\mathbb{i}\left(\frac{\pi}{4}+\theta\right)} \left(\Theta_n(E) - \Theta_n(W)\right) \tag{3.4}$$

holds, where N, S, W, E are the corners next to x according to Figure 3.3.

To prove this, we will show that a similar relation actually applies for each subgraph P in the summation in the definition of Θ separately. Let P_S be a subgraph used in the computation of $\Theta_n^{(\ell)}(S)$, where S is a southern corner. Let us define

$$R = \alpha^{|P_S|} e^{-\frac{\mathbb{i}}{2}\mathcal{W}(\pi(P_S))}(-1)^{\mathrm{loops}_{-2,2n}(P_S \setminus \pi(P_S))}(-1)^{\mathrm{sheet}(\pi(P_S))}.$$

Let us assume $x \notin P$, where P is a subgraph used in the computation of Θ_n. Since $\pi(P_S)$ ends at a south corner, we have $R \in \mathbb{R}$ as in Lemma 3.2. This is a contribution to the value of $\Theta_n^{(\ell)}(S)$, which we call $c_S = R$. Let us define a constant $\lambda = e^{\mathbb{i}\frac{\pi}{4}}$. We shall determine a contribution to $\Theta_n^{(\ell)}(W)$ in terms of R, where W is the west corner of x. The path will remain the same, except that it ends at a different angle causing the winding to be different by $\pi/2$. Therefore, the contribution to $\Theta_n^{(\ell)}(W)$ is $c_W = \overline{\lambda}R$.

When extending the path to the north corner, the length of the path increases by 1, and the final angle differs by π, so the contribution to $\Theta_n^{(\ell)}(N)$ is $c_N = -\mathbb{i}\alpha R$, where N is a north corner with respect to the edge x. Similarly, for the east corner the contribution is $c_E = \lambda\alpha R$.

Next, we show that the contributions satisfy the following relation:

$$(c_N - c_S) = e^{\mathbb{i}\left(\frac{\pi}{4}+\theta\right)} \left(c_E - c_W\right). \tag{3.5}$$

Plugging in the values of the contributions, we obtain

$$(-\mathbb{i}\alpha - 1)R$$

and as the left-hand side

$$e^{\mathbb{i}\left(\frac{\pi}{4}+\theta\right)}(\lambda\alpha - \overline{\lambda})R.$$

These are equal if and only if

$$\frac{1+\mathbb{i}\alpha}{1-\mathbb{i}\alpha} = e^{i\theta}.$$

Let us recall that we have $\alpha = e^{-2\beta^*} = \tan\frac{\theta}{2}$. Substituting this yields

$$\frac{1+\mathbb{i}\tan\frac{\theta}{2}}{1-\mathbb{i}\tan\frac{\theta}{2}} = \frac{1+\mathbb{i}\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}{1-\mathbb{i}\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}} = \cos^2\frac{\theta}{2} + 2\mathbb{i}\sin\frac{\theta}{2}\cos\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \cos\theta + \mathbb{i}\sin\theta = e^{\mathbb{i}\theta}.$$

Relation (3.5) can be derived for edges $x \in P$ using the same steps, taking into account how the length and winding of the path change between the four corners.

Taking the sum over all P and the limit $\ell \to \infty$ for both sides of (3.5), we obtain (3.4).

Using the same steps, it is also possible to derive the relation

$$\Theta_n(N') - \Theta(S') = e^{\mathbb{i}\left(\frac{3\pi}{4}-\theta\right)}\left(\Theta_n(E') - \Theta_n(W')\right) \tag{3.6}$$

for type \ edges. The computations are omitted here, as it would be repetition of the previous steps with slight changes.

Using relations (3.4) and (3.6), it is possible to solve the values of Θ_n at two of the corners next to an edge, provided that we know the values at the other two. For type / edges we use (3.4). Lemma 3.2 gives us subspaces for the complex numbers $\Theta_n(S)$, $\Theta_n(N)$, $\Theta_n(W)$, and $\Theta_n(E)$. Let r_S, r_N, r_W, and r_E be real numbers that satisfy

$$\Theta_n(S) = r_S, \qquad \Theta_n(N) = r_N\mathbb{i},$$
$$\Theta_n(W) = r_W e^{-\mathbb{i}\frac{\pi}{4}}, \qquad \Theta_n(E) = r_E e^{\mathbb{i}\frac{\pi}{4}}.$$

We write (3.4) as

$$r_N\mathbb{i} - r_S = e^{\mathbb{i}\theta}\left(r_E\mathbb{i} - r_W\right). \tag{3.7}$$

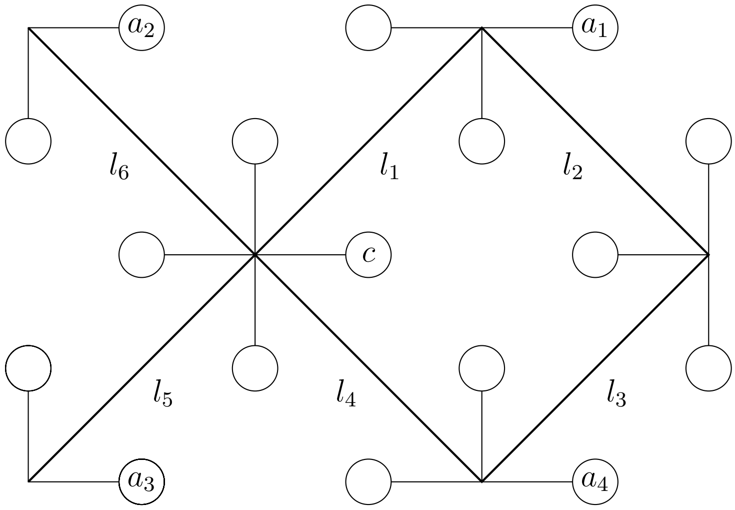

Figure 3.4: The edges and corners that are used to prove the massive harmonicity of Θ_n. To compute the value of $\Delta_\theta \Theta_n$ at c, we need to solve the values of Θ_n at the points a_1, a_2, a_3 and a_4 using the remaining unnamed corners in the figure.

Let

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

which rotates a vector by θ. Writing the real and imaginary parts of (3.7) separately gives us two equations that can be written in the form

$$\begin{pmatrix} -r_S \\ r_N \end{pmatrix} = R(\theta) \begin{pmatrix} -r_W \\ r_E \end{pmatrix}. \tag{3.8}$$

Let us assume that two of the numbers r_S, r_N, r_W, r_E are known. We can always solve the remaining two. In a similar manner we would derive a system of equations for edges of type \. That system is

$$\begin{pmatrix} r'_N \\ -r'_S \end{pmatrix} = R(-\theta) \begin{pmatrix} r'_W \\ r'_E \end{pmatrix}. \tag{3.9}$$

We use relations (3.8) and (3.9) for edges l_1, l_2, l_3, l_4, l_5, and l_6 in Figure 3.4. This allows us to write a system of twelve equations from which it is possible to solve expressions for the values at the points c, a_1, a_2, a_3 and a_4 as functions of the values in the remaining points drawn in Figure 3.4. Substituting these solutions to the formula of $\Delta_\theta \Theta_n$ at the point c yields 0. The computation is straightforward but tedious, so the technical details are omitted here.

When computing $\Delta_\theta \Theta_n(k + \mathbb{i}s - \frac{1}{2})$ near the branch cuts, where $k + \mathbb{i}s \in \{k, s | s = 0, k < -2 \text{ or } k > 2n\}$, three of the lines in Figure 3.4 cross the branch cut. Therefore $(-1)^{\text{sheet}}$ causes the values in the corners below the branch cut to have opposite sign compared to relations (3.4) and (3.6). Hence, the corresponding values in equations (3.8) and (3.9) have to be of the opposite sign, too. However, the definition of the operator Δ_θ has different signs for the corners on the other side of the branch cut, so the final result is still 0 in this case.

Proof of part 6. The proof of this part follows the same steps as the proof for the previous part, which we will not repeat here, but provide the general idea. This time, we have to carefully consider which edges in Figure 3.4 cross a branch cut and which do not. The corresponding equations have to be adjusted accordingly. We obtain a system of equations like we did in the previous part. The solutions of that equation are plugged into the definition of the operator Θ_θ, which completes the proof. For more information on this result, see [4] and [3]. □

Until now we have used Θ_n to denote a limit point of $\Theta_n^{(\ell)}$, but we have not had the certainty if the limit is unique. Theorem 3.1 showed that these limit points have some specific properties. The following lemma shows that the function defined on corners of the form $k + \mathbb{i}s - \frac{1}{2}$, satisfying some of these properties, is unique.

Lemma 3.3. *A function $F : \{k + \mathbb{i}s - \frac{1}{2} \mid k, s \in \mathbb{Z}\} \to \mathbb{C}$, which satisfies the conditions*

1. $|F(k + \mathbb{i}s - \frac{1}{2})| \leq C$
2. $\Delta_\theta F(k + \mathbb{i}s - \frac{1}{2}) = 0$ *for all* $k + \mathbb{i}s - \frac{1}{2} \notin \{0, 2n\}$
3. $\Delta_\theta F(-\frac{1}{2}) = \frac{1}{1+q^2} D^*_{n+1}$, $\Delta_\theta F(2n - \frac{1}{2}) = \frac{q^2}{1+q^2} D_{n+1}$
4. $F(-\frac{1}{2}) = D^*_n$, $F(2n - \frac{1}{2}) = D_n$, $F(k - \frac{1}{2}) = 0$ *for* $k \leq -2$ *or* $k \geq 2n+2$,

exists and is unique.

Proof. Let F be a function that satisfies the properties above. We know that such a function exists by Theorem 3.1. Let us assume that $\widetilde{F}$ is another solution. Let $G = F - \widetilde{F}$. Clearly $|G(k + \mathbb{i}s - \frac{1}{2})|$ is bounded from above as a sum of two bounded functions. Since the operator Δ_θ is linear, it is clear that $\Delta_\theta G(k + \mathbb{i}s - \frac{1}{2}) = 0$ for all k, s. Also, we have $G(k) = 0$ for all $k \leq 0$ and $k \geq 2n$. According to [3], the function G must then be constant. We know that there are points with $G(k - \frac{1}{2}) = 0$, so G is identically zero. Hence, $F = \widetilde{F}$ implying that the solution is unique. □

As a consequence of Lemma 3.3, if we only consider corners of the form $k + \mathbb{i}s - \frac{1}{2}$, the limiting function described in proposition 3.1 and Theorem 3.1 is unique, and we simply define

$$\Theta_n = \lim_{\ell \to \infty} \Theta_n^{(\ell)}.$$

regardless of the choice of the increasing sequence of graphs.

3.3 The Fourier Transform of Θ

Now that we have defined the function Θ_n, we will define its Fourier transform. It will turn out that the Fourier transform has some powerful properties that will enable us to show the connection between the spin-correlations and the orthogonal polynomials.

Definition 3.4. *Let us define the Fourier transform of Θ_n on the horizontal line at the level s, as*

$$\widehat{\Theta}_{n,s}(e^{\mathbb{i}t}) = \sum_{j \in J} e^{\mathbb{i}jt} \Theta_n(2j + \mathbb{i}s - \tfrac{1}{2}),$$

where

$$J = \begin{cases} \mathbb{Z} & \text{if } s \text{ even} \\ \mathbb{Z} + \frac{1}{2} & \text{if } s \text{ odd.} \end{cases}$$

We also use the shorthand

$$\widehat{\Theta}_n(e^{\mathbb{i}t}) = \widehat{\Theta}_{n,0}(e^{\mathbb{i}t}).$$

The following theorem shows that the Fourier transform of Θ_n has interesting properties. It is a trigonometric polynomial, and moreover it is lacking specific Fourier frequencies when multiplied by a certain weight function.

Theorem 3.2. *The Fourier transform $\widehat{\Theta}_n\left(e^{\mathbb{i}t}\right)$ is a trigonometric polynomial of the form*

$$\widehat{\Theta}_n\left(e^{\mathbb{i}t}\right) = D_n^* + \sum_{j=1}^{n-1} c_j^{(n)} e^{\mathbb{i}jt} + D_n e^{\mathbb{i}nt},$$

where $c_j^{(n)}$ are constants. Also, we have

$$\widehat{\Theta}_n\left(e^{\mathbb{i}t}\right) w\left(e^{\mathbb{i}t}\right) = \cdots + D_{n+1}^* + q^2 D_{n+1} e^{\mathbb{i}nt} + \cdots,$$

where $w\left(e^{\mathbb{i}t}\right) = (1+q^2)\left(1 - \left(m \cos \frac{t}{2}\right)^2\right)^{\frac{1}{2}}$ is called the weight function and $m = \sin(2\theta) = \frac{2}{q+q^{-1}}$ defines constants m and q with respect to the parameter θ.

Proof. The first part of the theorem follows from Definition 3.4 and the identities $\Theta_n(-\frac{1}{2}) = D_n^*, \Theta_n(2n - \frac{1}{2}) = D_n$ and $\Theta_n(k - \frac{1}{2}) = 0$ for all $k < 0$ and $k > 2n$ from Theorem 3.1.

For the second part, we assume $s > 0$, and use the property stating $\Delta_\theta \Theta_n(k + \mathbb{i}s - \frac{1}{2}) = 0$ from Theorem 3.1. We obtain

$$\begin{aligned}\Theta_n(k + \mathbb{i}s - \tfrac{1}{2}) - \frac{1}{4}\sin(2\theta)\cdot \big[&\Theta_n((k+1) + \mathbb{i}(s+1) - \tfrac{1}{2}) \\ &+ \Theta_n((k+1) + \mathbb{i}(s-1) - \tfrac{1}{2}) \\ &+ \Theta_n((k-1) + \mathbb{i}(s+1) - \tfrac{1}{2}) \\ &+ \Theta_n((k-1) + \mathbb{i}(s-1) - \tfrac{1}{2})\big] = 0.\end{aligned}$$

We multiply both sides of this equation by $e^{\frac{1}{2}\mathbb{i}kt}$ and sum over all $k \in 2\mathbb{Z}$, which results in the equation

$$\begin{aligned}\widehat{\Theta}_{n,s}\left(e^{\mathbb{i}t}\right) - \frac{1}{4}\sin(2\theta)\Big[&e^{-\frac{1}{2}\mathbb{i}t}\widehat{\Theta}_{n,s+1}\left(e^{\mathbb{i}t}\right) + e^{-\frac{1}{2}\mathbb{i}t}\widehat{\Theta}_{n,s-1}\left(e^{\mathbb{i}t}\right) \\ &+ e^{\frac{1}{2}\mathbb{i}t}\widehat{\Theta}_{n,s+1}\left(e^{\mathbb{i}t}\right) + e^{\frac{1}{2}\mathbb{i}t}\widehat{\Theta}_{n,s-1}\left(e^{\mathbb{i}t}\right)\Big] = 0.\end{aligned}$$

Let us note that we have $e^{-\frac{1}{2}\mathbb{i}t} + e^{\frac{1}{2}\mathbb{i}t} = 2\cos\frac{t}{2}$ and $\sin(2\theta) = m$. Using these we write

$$\widehat{\Theta}_{n,s}\left(e^{\mathbb{i}t}\right) - \frac{m}{2}\cos\frac{t}{2}\left(\widehat{\Theta}_{n,s+1}\left(e^{\mathbb{i}t}\right) + \widehat{\Theta}_{n,s-1}\left(e^{\mathbb{i}t}\right)\right) = 0,$$

that is a recurrence relation with respect to s. By defining $a_s = \widehat{\Theta}_{n,s}\left(e^{\mathbb{i}t}\right)$ and $M = \frac{m}{2}\cos\frac{t}{2}$, we re-write the relation as

$$a_s - M(a_{s+1} + a_{s-1}) = 0, \; s \geq 1. \tag{3.10}$$

In order to solve the recurrence relation, we use $a_s = r^s$ as a trial function. Then we have

$$r^s - M\left(r^{s+1} + r^{s-1}\right) = 0,$$

and moreover

$$r^2 - \frac{1}{M}r + 1 = 0.$$

The quadratic equation has roots

$$r_- = \frac{1 - \sqrt{1 - 4M^2}}{2M}, \qquad r_+ = \frac{1 + \sqrt{1 - 4M^2}}{2M}.$$

Hence the general solution of (3.10) is

$$a_s = A_+ r_+^s + A_- r_-^s$$

for some constants A_+ and A_-. Since we defined $M = \frac{1}{4}\sin(2\theta)$, we know that $0 < r_- < 1$ and $r_+ > 1$. We also know that a_s is bounded [3]. Thus we require $A_+ = 0$, as otherwise the solution would not be bounded. Hence, the unique solution is

$$a_s = a_0 r_-^s,$$

that is

$$\widehat{\Theta}_{n,s}\left(e^{\mathbb{i}t}\right) = \left(\frac{1-\sqrt{1-\left(m\cos\frac{t}{2}\right)^2}}{m\cos\frac{t}{2}}\right)^s \widehat{\Theta}_{n,0}\left(e^{\mathbb{i}t}\right).$$

Let us consider the value of $\Delta_\theta\Theta_n(k-\frac{1}{2})$, $(s=0)$. Theorem 3.1 states

$$\Delta_\theta\Theta_n(k-\frac{1}{2}) = \begin{cases} \frac{1}{1+q^2}D^*_{n+1} & \text{if } k=0 \\ \frac{q^2}{1+q^2}D_{n+1} & \text{if } k=2n \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, we have

$$\begin{aligned} \Theta_n(k-\tfrac{1}{2}) - \frac{1}{4}\sin(2\theta)\cdot \big[&\Theta_n((k+1)+\mathbb{i}-\tfrac{1}{2}) + \Theta_n((k+1)-\mathbb{i}-\tfrac{1}{2}) + \\ &\Theta_n((k-1)+\mathbb{i}-\tfrac{1}{2}) + \Theta_n((k-1)-\mathbb{i}-\tfrac{1}{2})\big] \\ &= \begin{cases} \frac{1}{1+q^2}D^*_{n+1} & \text{if } k=0 \\ \frac{q^2}{1+q^2}D_{n+1} & \text{if } k=2n \\ 0 & \text{if } 0<k<2n. \end{cases} \end{aligned} \tag{3.11}$$

If $k < 0$ or $k > 2$, Theorem 3.1 does not tell us anything about the value of the above expression, so the values at these points might be zero or non-zero. Note that the left-hand side is equal to $\Delta_\theta\Theta_n(k-\frac{1}{2})$ only when the point in question is far enough from the branch cuts, which is why the right-hand side is allowed to be non-zero near them.

We multiply both sides of (3.11) by $e^{\frac{1}{2}\mathbb{i}kt}$ and sum over $k \in 2\mathbb{Z}$ as we did before. The left-hand side becomes

$$\widehat{\Theta}_{n,0}\left(e^{\mathbb{i}t}\right) - \frac{m}{4}\cdot\left[e^{-\frac{1}{2}\mathbb{i}kt}\widehat{\Theta}_{n,1}\left(e^{\mathbb{i}t}\right) + e^{-\frac{1}{2}\mathbb{i}kt}\widehat{\Theta}_{n,-1}\left(e^{\mathbb{i}t}\right) e^{\frac{1}{2}\mathbb{i}kt}\widehat{\Theta}_{n,1}\left(e^{\mathbb{i}t}\right) e^{\frac{1}{2}\mathbb{i}kt}\widehat{\Theta}_{n,-1}\left(e^{\mathbb{i}t}\right)\right].$$

Let us use identities $e^{-\frac{1}{2}\mathbb{i}t} + e^{\frac{1}{2}\mathbb{i}t} = 2\cos\frac{t}{2}$ and $\widehat{\Theta}_{n,1}\left(e^{\mathbb{i}t}\right) = \widehat{\Theta}_{n,-1}\left(e^{\mathbb{i}t}\right)$ to obtain a simpler form

$$
\begin{aligned}
&\widehat{\Theta}_{n,0}\left(e^{\mathrm{i}t}\right) - m\cos\frac{t}{2}\widehat{\Theta}_{n,1}\left(e^{\mathrm{i}t}\right)\\
=&\widehat{\Theta}_{n,0}\left(e^{\mathrm{i}t}\right) - m\cos\left(\frac{t}{2}\right)\frac{1-\sqrt{1-\left(m\cos\frac{t}{2}\right)^2}}{m\cos\frac{t}{2}}\widehat{\Theta}_{n,0}\left(e^{\mathrm{i}t}\right)\\
=&\widehat{\Theta}_{n,0}\left(e^{\mathrm{i}t}\right)\sqrt{1-\left(m\cos\frac{t}{2}\right)^2}.
\end{aligned}
$$

After summation and multiplication by $e^{\frac{1}{2}\mathrm{i}kt}$, the right-hand side of (3.11) becomes a series of the form

$$
\cdots + \frac{1}{1+q^2}D^*_{n+1} + \frac{q^2}{1+q^2}D_{n+1}e^{\mathrm{i}nt} + \cdots,
$$

that is lacking the terms of degrees 1 to $n-1$. By combining this with the left-hand side, we have

$$
\widehat{\Theta}_{n,0}\left(e^{\mathrm{i}t}\right)\sqrt{1-\left(m\cos\frac{t}{2}\right)^2} = \cdots + \frac{1}{1+q^2}D^*_{n+1} + \frac{q^2}{1+q^2}D_{n+1}e^{\mathrm{i}nt} + \cdots.
$$

Multiplying both sides by $1+q^2$ completes the proof. $\square$

Theorem 3.2 explicitly shows the connection between the Ising model and the orthogonal polynomials on the unit circle. As we apply operation $(\cdot \mapsto \frac{1}{2\pi}\int_0^{2\pi} \cdot\, e^{-\mathrm{i}jt}\mathrm{d}t)$ for the both sides of the second identity in Theorem 3.2, we see that the Fourier transform $\widehat{\Theta}$ satisfies the orthogonality condition

$$
\frac{1}{2\pi}\int_0^{2\pi}\widehat{\Theta}_n\left(e^{\mathrm{i}t}\right)e^{-\mathrm{i}jt}w\left(e^{\mathrm{i}t}\right)\mathrm{d}t = \begin{cases} 0, & j = 1,\cdots,n-1\\ D^*_{n+1}, & j=0\\ q^2D_{n+1}, & j=n\end{cases}. \tag{3.12}
$$

In the following chapters we will study orthogonal polynomials. At the critical temperature, we will show that $\widehat{\Theta_n}$ has an explicit representation. In the subcritical temperature, we do not have an explicit representation, but we can analyze the situation with the help of the theory of the orthogonal polynomials.

Chapter 4

The Spin Correlation Function at the Critical Temperature

In Chapter 3 we defined the function $\widehat{\Theta}_n$ and proved some of its properties that connect it to the diagonal spin correlations D_n and D_n^*. In this chapter, we will show that the function $\widehat{\Theta}_n$ is proportional to the function $e^{\mathrm{i}\frac{nt}{2}} L_n(\cos \frac{t}{2})$, where L_n is the Legendre polynomial of degree n. Then we use this function to prove the following theorem concerning the spin correlation function at the critical temperature, which is the main result of this chapter.

Theorem 4.1. *At the critical temperature $\beta = \beta_{crit}$, the diagonal spin correlation function is*

$$D_n = \left(\frac{2}{\pi}\right)^n \prod_{k=1}^{n-1} \left(1 - \frac{1}{4k^2}\right)^{k-n} \sim 2^{1/3} e^{3\zeta'(-1)} (2n)^{-1/4}.$$

The proof will be at the end of this chapter. The notation $f(n) \sim g(n)$ means that we have $\lim_n f(n)/g(n) = 1$. The exact formula was first published in [22].

4.1 Legendre Polynomials

The explicit formula of the function $\widehat{\Theta}_n$ will contain the Legendre polynomial of degree n. We will define the Legendre polynomials, and then we will prove some basic properties they have that will be needed in the later analysis.

The Legendre polynomials appear in the literature with different normalizations depending on the context. We use the monic Legendre polynomial with the following definition.

Definition 4.1. *The monic Legendre polynomial* $L_n : \mathbb{R} \to \mathbb{R}$ *of degree* $n \in \mathbb{N}$ *is defined as*

$$L_n(x) = \frac{n!}{(2n)!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n.$$

The following lemma confirms that the object we defined really is a polynomial of the degree n, and that it is indeed monic, i.e. its leading coefficient equals 1.

Lemma 4.1. *The function* $L_n(x)$ *is a monic polynomial of degree* n*, i.e. it is of the form*

$$L_n(x) = x^n + \sum_{k=1}^{n-1} c_k^{(n)} x^k.$$

Proof. Clearly, L_n is a polynomial since $(x^2-1)^n$ is a degree $2n$ polynomial, and differentiating n times yields a degree n polynomial. Moreover, after differentiating the expression $(x^2-1)^n$ n times, the coefficient of the leading term is $2n \cdot (2n-1) \cdots (n+1) = (2n)!/n!$. Thus the leading coefficient of the polynomial $L_n(x)$ is always 1, i.e. $L_n(x) = x^n + \sum_{k=0}^{n-1} c_k^{(n)} x^k$. $\square$

The following lemma is an auxiliary result that we will use later when analyzing the orthogonal properties of the Legendre polynomials.

Lemma 4.2. *We have*

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}(x^2-1)^n = 0,$$

for all $x = \pm 1$ *and* $m = 0, 1, \cdots, (n-1)$.

Proof. Let us apply the Leibniz rule, which yields

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}(x^2-1)^n = \frac{\mathrm{d}^m}{\mathrm{d}x^m}\prod_{l=1}^{n}(x^2-1) \tag{4.1}$$

$$= \sum_{k_1+k_2+\cdots+k_n=m} \frac{m!}{k_1!k_2!\cdots k_n!}\prod_{l=1}^{n}\frac{\mathrm{d}^{k_l}}{\mathrm{d}x^{k_l}}(x^2-1) \tag{4.2}$$

The sum of the numbers k_l in (4.2) is m, so some of them have to be 0 since $n > m$. This means that the product has a factor x^2-1 for every term in the sum. Thus, the entire expression has the value 0, as $x = \pm 1$. $\square$

The following is a technical result, that will be important later when determining the L^2-norm of the Legendre polynomial.

Lemma 4.3. *The identity*

$$\int_{-1}^{1} (x^2 - 1)^n \mathrm{d}x = (-1)^n \frac{2^{2n+1}(n!)^2}{(2n+1)!}$$

holds for all $n \in \mathbb{N}$.

Proof. Let us denote

$$I_n = \int_{-1}^{1} (x^2 - 1)^n \mathrm{d}x.$$

We assume that $n \geq 1$. We integrate by parts, after which we rearrange the terms in order to obtain a recursive equation, which yields

$$\begin{aligned}
I_n &= \int_{-1}^{1} 1 \cdot (x^2 - 1)^n \mathrm{d}x = -2n \int_{-1}^{1} x^2 (x^2 - 1)^{n-1} \mathrm{d}x \\
&= -2n \left[\int_{-1}^{1} (x^2 - 1)^{n-1} \mathrm{d}x + \int_{-1}^{1} (x^2 - 1)^n \mathrm{d}x \right] \\
&= -2n(I_{n-1} + I_n).
\end{aligned}$$

Here, we can solve

$$I_n = -\frac{2n}{2n+1} I_{n-1}.$$

It is trivial from the definition that $I_0 = 2$. Next we express I_n as a function of the known case I_0 and perform a substitution $I_0 = 2$, which yields

$$I_n = \left(\prod_{k=1}^{n} -\frac{2k}{2k+1} \right) I_0 = (-1)^n \frac{2^{2n+1}(n!)^2}{(2n+1)!}.$$

□

The following proposition shows that the Legendre polynomial of degree n is orthogonal with respect to monomials x^m, where $0 \leq m < n$. As a corollary, we get the result that Legendre polynomials are orthogonal polynomials, and we also derive an expression for the L^2 norms of the Legendre polynomials.

Note that in this work we define $0!! = (-1)!! = 1$.

Proposition 4.1. *The orthogonality condition*

$$\int_{-1}^{1} L_n(x) x^m dx = 0$$

applies for all $n \geq 1$ *and* $m = 0, \cdots, (n-1)$. *Moreover, we have*

$$\int_{-1}^{1} L_n(x) x^n \mathrm{d}x = \frac{2}{2n+1} \cdot \left(\frac{n!}{(2n-1)!!} \right)^2.$$

Proof. Integration by parts gives us

$$\int_{-1}^{1} L_n(x)x^m dx = \frac{n!}{(2n)!}\int_{-1}^{1}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[(x^2-1)^n\right]x^m\mathrm{d}x$$
$$= \frac{n!}{(2n)!}\Big|_{-1}^{1}\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}\left[(x^2-1)^n\right]x^m - \frac{n!}{(2n)!}\int_{-1}^{1}\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}\left[(x^2-1)^n\right]mx^{m-1}\mathrm{d}x.$$

The first term in the last expression has value 0 by Lemma 4.2. Integration by parts repeatedly yields

$$\begin{aligned}
&-m\int_{-1}^{1}\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}\left[(x^2-1)^n\right]x^{m-1}\mathrm{d}x\\
=&(-m)(-(m-1))\int_{-1}^{1}\frac{\mathrm{d}^{n-2}}{\mathrm{d}x^{n-2}}\left[(x^2-1)^n\right]x^{m-2}\mathrm{d}x\\
=&\cdots\\
=&(-1)^m m!\int_{-1}^{1}\frac{\mathrm{d}^{n-m}}{\mathrm{d}x^{n-m}}\left[(x^2-1)^n\right]x^0\mathrm{d}x. \qquad (4.3)
\end{aligned}$$

Assume that $m \leq n-1$. Integration and Lemma 4.2 yield

$$(-1)^m m!\Big|_{-1}^{1}\frac{\mathrm{d}^{n-m-1}}{\mathrm{d}x^{n-m-1}}\left[(x^2-1)^n\right] = 0.$$

If we have $m = n$, then the expression $(x^2-1)^n$ in (4.3) is not differentiated, so the integral has the form

$$(-1)^n n!\int_{-1}^{1}(x^2-1)^n\mathrm{d}x.$$

We use the result from Lemma 4.3, and obtain

$$\begin{aligned}
\int_{-1}^{1} L_n(x)x^n\mathrm{d}x &= (-1)^n n!\frac{n!}{(2n)!}(-1)^n\frac{2^{2n+1}(n!)^2}{(2n+1)!}\\
&= \frac{2}{2n+1}\cdot\left(\frac{n!}{(2n-1)!!}\right)^2.
\end{aligned}$$

□

Corollary 4.1. *Legendre polynomials are orthogonal with respect to the Lebesque measure on the interval* $[-1,1]$, *that is*

$$\int_{-1}^{1} L_n(x)L_m(x)\mathrm{d}x = 0, \text{ if } n \neq m.$$

Moreover, the square of the L^2 norm is

$$\int_{-1}^{1} L_n(x)^2 \mathrm{d}x = \frac{2}{2n+1} \cdot \left(\frac{n!}{(2n-1)!!} \right)^2 \tag{4.4}$$

Proof. We apply the knowledge that L_m is a polynomial of degree m, and Proposition 4.3. We write

$$\begin{aligned} \int_{-1}^{1} L_n(x) L_m(x) \mathrm{d}x &= \int_{-1}^{1} L_n(x)(x^m + c_{m-1}^{(m)} x^{m-1} + \cdots) \mathrm{d}x \\ &= \int_{-1}^{1} L_n(x) x^m dx + c_{m-1}^{(m)} \int_{-1}^{1} L_n(x) x^{m-1} \mathrm{d}x + \cdots . \end{aligned}$$

Let us assume that $m < n$ and apply Lemma 4.3, and we see that each term in the sum is 0. On the other hand, if $m = n$, the first term in the sum is non-zero, and its value can be obtained directly using Lemma 4.3. □

The following result will be a part of an argument later on. Let us recall that a function f is called even if $f(x) = f(-x)$ and odd if $f(x) = -f(-x)$ for all x.

Lemma 4.4. *The Legendre polynomial $L_n(x)$ is even if n is even. $L_n(x)$ is odd if n is odd.*

Proof. Scalar multiplication clearly does not affect the parity of a function, so it is sufficient to study the derivative

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n.$$

The function to be differentiated is even. The derivative of an even function is odd, and the derivative of an odd function is even. Thus, differentiating an even function n times results in an even function if n is even, and in an odd function if n is odd. □

4.2 Computing the Correlation Function

At the critical temperature $\beta = \beta_{\mathrm{crit}}$, we have $\beta = \beta^*$, and $D_n = D_n^*$. Also, we have $q = m = 1$, which means that the weight function in Theorem 3.2 simplifies to $w\left(e^{\mathbb{i}t}\right) = 2|\sin\frac{t}{2}|$. Thus, we know that the function $\widehat{\Theta}_n$ satisfies

$$\widehat{\Theta}_n(e^{\mathbb{i}t}) = D_n + \cdots + D_n e^{\mathbb{i}nt} \tag{4.5}$$

and

$$2\int_0^{2\pi} \widehat{\Theta}_n(e^{\mathbb{i}t})e^{-\mathbb{i}jt}\sin\tfrac{t}{2}\frac{\mathrm{d}t}{2\pi} = \begin{cases} D_n, & \text{if } j\in\{0,n\}\\ 0, & \text{if } j=1,\cdots,n-1. \end{cases}$$

We will show that the polynomial $A_n e^{\mathbb{i}\frac{nt}{2}} L_n\left(\cos\frac{t}{2}\right)$ satisfies these conditions with some constant A_n.

Lemma 4.5. *We have*

$$e^{\mathbb{i}\frac{nt}{2}} L_n(\cos\tfrac{t}{2}) = \sum_{k=0}^{n} a_k e^{\mathbb{i}kt},$$

where c_k are some constants and specifically $a_0 = a_n = 2^{-n}$.

Proof. The left-hand side can be written as

$$e^{\mathbb{i}\frac{nt}{2}} \sum_{j=0}^{n} c_j^{(n)} \frac{1}{2^j}\left(e^{\mathbb{i}\frac{t}{2}} + e^{-\mathbb{i}\frac{t}{2}}\right)^j = \sum_{j=0}^{n} c_j^{(n)} \frac{1}{2^j}\sum_{k=0}^{j}\binom{j}{k} e^{\mathbb{i}\frac{t}{2}(2k-j+n)}.$$

According to Lemma 4.4, the function L_n is even for an even n and odd for an odd n. Hence, if n is even, then we have $c_j^{(n)} = 0$ for odd values of j, and if n is odd, then $c_j^{(n)} = 0$ for even values of j. Thus, the number $2k-j+n$ is always even for non-zero k, j, n. This implies that the above expression is of the form $\sum_{k=0}^{n} a_k e^{\mathbb{i}kt}$.

The term of the maximal degree $e^{\mathbb{i}nt}$ is produced only if $j = k = n$. We know that $c_n^{(n)} = 1$ due to L_n being monic, so $a_n = 2^{-n}$. The constant term of degree 0 is produced only if $j = n$ and $k = 0$. By substituting these to the above formula, we obtain $a_0 = 2^{-n}$. □

The following proposition shows that the function $e^{\mathbb{i}\frac{nt}{2}} L_n\left(\cos\frac{t}{2}\right)$ is orthogonal to the monomials $e^{\mathbb{i}t}, \cdots, e^{\mathbb{i}(n-1)t}$.

Proposition 4.2. *For all $j = 1, ..., (n-1)$, we have the orthogonality condition*

$$\int_0^{2\pi} e^{\mathbb{i}\frac{nt}{2}} L_n\left(\cos\frac{t}{2}\right) e^{-\mathbb{i}jt}\sin\frac{t}{2}\mathrm{d}t = 0.$$

Proof. We apply Euler's formula and write the exponential functions, using the sine and cosine functions, as

$$\begin{aligned} e^{\mathbb{i}\frac{t}{2}} &= \cos\frac{t}{2} + \mathbb{i}\sin\frac{t}{2},\\ e^{\mathbb{i}\frac{nt}{2}} &= \left(e^{\mathbb{i}\frac{t}{2}}\right)^n = \left(\cos\frac{t}{2} + \mathbb{i}\sin\frac{t}{2},\right)^n\\ e^{-\mathbb{i}jt} &= \left(e^{\mathbb{i}\frac{t}{2}}\right)^{-2j} = \left(\cos\frac{t}{2} + \mathbb{i}\sin\frac{t}{2}\right)^{-2j}. \end{aligned}$$

With the help of these, we write the integral in the form

$$S_{n,j} = \frac{1}{2}\int_0^{2\pi} \left(\cos\frac{t}{2} + \mathbb{i}\sin\frac{t}{2}\right)^{n-2j} L_n\left(\cos\frac{t}{2}\right)\sin\frac{t}{2}\mathrm{d}t.$$

Let us perform a change of variables $x = \cos\frac{t}{2}$. Then, for the integration we have $dx = -\frac{1}{2}\sin\frac{t}{2}dt$, and the new limits of integration are $2\arccos 0 = 1$ and $2\arccos(2\pi/2) = -1$. We also notice that we have $\sin(\frac{t}{2}) = \sqrt{1-x^2}$ for $t \in [0, 2\pi]$. We apply these, which yields

$$S_{n,j} = \int_{-1}^{1} \left(x + \mathbb{i}\sqrt{1-x^2}\right)^{n-2j} L_n(x)\mathrm{d}x.$$

The objective is to show that $S_{n,j} = 0$.

The numbers $z^m = \left(x + \mathbb{i}\sqrt{1-x^2}\right)^m$ are points on the unit circle, so they can be expressed as $z^m = e^{\mathbb{i}\phi m}$, and the corresponding complex conjugate is $\overline{z^m} = \overline{e^{\mathbb{i}\phi m}} = e^{-\mathbb{i}\phi m} = (e^{\mathbb{i}\phi})^{-m} = z^{-m}$. Hence, changing the sign of the exponent has no effect on the real part of the number. $L_n(x)$ is real for all $x \in \mathbb{R}$, so changing the sign of the exponent does not change the value of the real part of the integral. The imaginary part remains the same except its sign. Thus, without loss of generality, we can limit our analysis to the cases that satisfy $n - 2j \geq 0$.

Let us assume that $n - 2j \geq 0$. Then the expression can be written as

$$S_{n,j} = \sum_{k=0}^{n-2j} \binom{n-2j}{k} \mathbb{i}^k \int_{-1}^{1} \sqrt{1-x^2}^k x^{n-2j-k} L_n(x)\mathrm{d}x$$

using Newton's binomial formula.

We analyze the real and imaginary parts separately. The real part is

$$\operatorname{Re} S_{n,j} = \sum_{k=0}^{n-2j} \binom{n-2j}{k} \operatorname{Re}\left(\mathbb{i}^k\right) \int_{-1}^{1} \sqrt{1-x^2}^k x^{n-2j-k} L_n(x)\mathrm{d}x.$$

Due to the factor $\operatorname{Re}\left(\mathbb{i}^k\right)$, all the terms with an odd index in the sum clearly have value 0. What remains is to show, what happens with even values of k. We will change variables, $k = 2l$. The expression can be written using l in the form

$$\operatorname{Re} S_{n,j} = \sum_{l=0}^{\lfloor \frac{n}{2} - j \rfloor} \binom{n-2j}{2l} (-1)^l \int_{-1}^{1} \left(1-x^2\right)^l x^{n-2j-2l} L_n(x)\mathrm{d}x.$$

The polynomial $L_n(x)$ is multiplied by the expression $\left(1-x^2\right)^l x^{n-2j-2l}$ inside the integral. The latter is a polynomial of degree $n-2j<n$. According to Proposition 4.1, the integral has value 0, which implies that

$$\operatorname{Re} S_{n,j} = 0. \tag{4.6}$$

Next, we examine the imaginary part

$$\operatorname{Im} S_{n,j} = \sum_{k=0}^{n-2j} \binom{n-2j}{k} \operatorname{Im}\left(\mathbb{i}^k\right) \int_{-1}^{1} \sqrt{1-x^2}^k x^{n-2j-k} L_n(x)\mathrm{d}x.$$

First, we note that $\operatorname{Im}\left(\mathbb{i}^k\right)=0$, if k is even. We study closer the case when k is odd. Let us consider the parity of the integrand $\sqrt{1-x^2}^k x^{n-2j-k} L_n(x)$. The function $x \mapsto \sqrt{1-x^2}^k$ is even, and the function $x \mapsto x^{n-2j-k}$ is even if n is odd. The function L_n is even, if n is even, otherwise it is odd. The product of an even and an odd function is an odd function. Thus $x^{n-2j-k}L_n(x)$ is odd, since always one of the factors is even and one is odd. This implies that the expression $\sqrt{1-x^2}^k x^{n-2j-k} L_n(x)$ is odd as a whole. Integrating an odd function over a symmetric interval yields 0, and we have

$$\operatorname{Im} S_{n,j} = 0. \tag{4.7}$$

Combining the results (4.6) and (4.7) gives us the desired result

$$S_{n,j} = 0.$$

□

The following lemma shows that the function we have been studying is the same as $\widehat{\Theta}_n$. The proof of the lemma uses Corollary 5.1, that is more convenient to prove once we have developed certain tools from the theory of the orthogonal polynomials on the unit circle in Chapter 5. Hence we postpone the proof of Corollary 5.1 to Chapter 5.

Lemma 4.6. *The following holds:*

$$\widehat{\Theta}_n(e^{\mathbb{i}t}) = 2^n D_n e^{\mathbb{i}\frac{nt}{2}} L_n(\cos\tfrac{t}{2}).$$

Proof. Let us define the function

$$P_n\left(e^{\mathbb{i}t}\right) = \widehat{\Theta}_n\left(e^{\mathbb{i}t}\right) - 2^n D_n e^{\mathbb{i}\frac{t}{2}} L_n(\cos\tfrac{t}{2}).$$

Proving the claim is equivalent to showing that we have

$$P_n(z) = 0$$

for all $z \in \mathbb{C}$.

Lemma 4.5 and (4.5) imply that

$$P_n\left(e^{\mathbb{i}t}\right) = \sum_{j=1}^{n-1} a_j e^{\mathbb{i}jt}$$

for some coefficients a_j. At this point we know that P_n is a polynomial of the degree $n-1$ at most, and we have $P_n(0) = 0$. Proposition 4.2 and (3.12) imply that this polynomial is orthogonal to the monomials $e^{\mathbb{i}t}, \cdots, e^{\mathbb{i}(n-1)t}$ with respect to the weight $w(e^{\mathbb{i}t}$. According to Corollary 5.1, we have $P_n(z) = 0$. □

Lemma 4.7. *The ratio of subsequent spin correlations satisfies*

$$\frac{D_{n+1}}{D_n} = \frac{2^{2n}}{\pi} \cdot \frac{2}{2n+1} \cdot \left(\frac{n!}{(2n-1)!!}\right)^2$$

for all $n \geq 0$.

Proof. With the help of the change of variables $x = \cos\frac{t}{2}$, we can write

$$2^n D_n \int_{-1}^{1} L_n(x) x^n \mathrm{d}x = \frac{1}{2} \cdot 2^n D_n \int_0^{2\pi} L_n\left(\cos\tfrac{t}{2}\right)\left(\cos\tfrac{t}{2}\right)^n \sin\tfrac{t}{2} \mathrm{d}t.$$

According to Lemma 4.6 we have $\widehat{\Theta}_n(e^{\mathbb{i}t}) = 2^n D_n e^{\mathbb{i}\frac{nt}{2}} L_n(\cos\frac{t}{2})$. Also, we have $w\left(e^{\mathbb{i}t}\right) = 2|\sin\frac{t}{2}|$. Therefore, the expression above can be written as

$$\begin{aligned}
&\frac{1}{4}\int_0^{2\pi} \widehat{\Theta}_n\left(e^{\mathbb{i}t}\right) w\left(e^{\mathbb{i}t}\right)\left(\cos\tfrac{t}{2}\right)^n \mathrm{d}t \\
=&\frac{1}{4}\int_0^{2\pi} (\cdots + D^*_{n+1} + D_{n+1}e^{\mathbb{i}nt} + \cdots)(\cos\tfrac{t}{2})^n e^{-\mathbb{i}\frac{nt}{2}} \mathrm{d}t \\
=&\frac{1}{4}\cdot\frac{1}{2^n}\int_0^{2\pi} (\cdots + D^*_{n+1} + D_{n+1}e^{\mathbb{i}nt} + \cdots)(1+e^{-\mathbb{i}t})^n \mathrm{d}t.
\end{aligned}$$

Using the binomial formula, this can be expressed as

$$\frac{1}{4}\cdot\frac{1}{2^n}\sum_{j=0}^{n}\binom{n}{j}\int_0^{2\pi}\left(\cdots + D_{n+1} + D_{n+1}e^{\mathbb{i}nt} + \cdots\right)e^{-\mathbb{i}jt}\mathrm{d}t.$$

Due to the orthogonality of the monomials $e^{\mathbb{i}jt}$, the only non-zero terms in the sum above are when $j = 0$ or $j = n$. Thus the sum equals $4\pi D_{n+1}$ and the entire expression simplifies to

$$\frac{\pi}{2^n} D_{n+1}.$$

On the other hand, using Proposition 4.1 we obtain

$$2^n D_n \int_{-1}^{1} L_n(x) x^n \mathrm{d}x = 2^n D_n \frac{2}{2n+1} \cdot \left(\frac{n!}{(2n-1)!!}\right)^2.$$

Now we solve the ratio

$$\frac{D_{n+1}}{D_n} = \frac{2^{2n}}{\pi} \cdot \frac{2}{2n+1} \cdot \left(\frac{n!}{(2n-1)!!}\right)^2.$$

□

At this point we have all the necessary results to complete the proof of Theorem 4.1, that gives an explicit formula for the spin correlation function at the critical temperature.

Proof of Theorem 4.1. Lemma 4.7 gives us the ratio between consecutive spin correlations, that is

$$\begin{aligned} \frac{D_{n+1}}{D_n} &= \frac{2^{2n}}{\pi} \cdot \frac{2}{2n+1} \cdot \left(\frac{n!}{(2n-1)!!}\right)^2 \\ &= \frac{2}{\pi} \cdot 2^{2n} \cdot \frac{n!}{(2n-1)!!} \cdot \frac{n!}{(2n+1)!!}. \end{aligned}$$

We know that $D_0 = 1$ as it is simply the square of a ± 1-valued random variable. Let us solve D_n using the telescope product

$$\begin{aligned} D_n = \frac{D_n}{D_0} &= \prod_{k=0}^{n-1} \frac{D_{k+1}}{D_k} \\ &= \prod_{k=0}^{n-1} \frac{2}{\pi} \cdot 2^{2k} \cdot \frac{k!}{(2k-1)!!} \cdot \frac{k!}{(2k+1)!!}. \end{aligned}$$

The value of the product can be computed part by part. First, we have

$$\prod_{k=0}^{n-1} \frac{2}{\pi} = \left(\frac{2}{\pi}\right)^n. \tag{4.8}$$

Next, we compute

$$\prod_{k=0}^{n-1} 2^{2k} = 2^{2\sum_{k=0}^{n-1} k} = 2^{n^2-n}. \tag{4.9}$$

Consider the term $\frac{k!}{(2k-1)!!}$. The factorials can be expressed using products of appropriate powers, which yields

$$\begin{aligned}\prod_{k=0}^{n-1}\frac{k!}{(2k-1)!!} &= \frac{1^n\cdot 1^{n-1}\cdot 2^{n-2}\cdots(n-2)^2\cdot(n-1)^1}{1^n\cdot 1^{n-1}\cdot 3^{n-2}\cdots(2n-5)^2\cdot(2n-3)^1}\\ &=\prod_{k=1}^{n-1}\frac{k^{n-k}}{(2k-1)^{n-k}}=\prod_{k=1}^{n-1}\left(2\cdot\left(1-\frac{1}{2k}\right)\right)^{k-n}\\ &=2^{-\frac{1}{2}(n^2-n)}\prod_{k=1}^{n-1}\left(1-\frac{1}{2k}\right)^{k-n}.\end{aligned}$$

The term $\frac{k!}{(2k+1)!!}$ can be simplified with a similar idea:

$$\begin{aligned}\prod_{k=0}^{n-1}\frac{k!}{(2k+1)!!} &= \frac{1^n\cdot 1^{n-1}\cdot 2^{n-2}\cdots(n-2)^2\cdot(n-1)^1}{1^n\cdot 3^{n-1}\cdots(2n-2)^2\cdot(2n-1)^1}\\ &=\prod_{k=1}^{n-1}\frac{k^{n-k}}{(2k+1)^{n-k}}=2^{-\frac{1}{2}(n^2-n)}\prod_{k=1}^{n-1}\left(1+\frac{1}{2k}\right)^{k-n}.\end{aligned}$$

Using the above representations, we write D_n as

$$\begin{aligned}D_n &= \left(\frac{2}{\pi}\right)^n\cdot 2^{n^2-n}\cdot 2^{-\frac{1}{2}(n^2-n)}\left(\prod_{k=1}^{n-1}\left(1-\frac{1}{2k}\right)^{k-n}\right)\\ &\quad\cdot 2^{-\frac{1}{2}(n^2-n)}\prod_{k=1}^{n-1}\left(1+\frac{1}{2k}\right)^{k-n}\\ &=\left(\frac{2}{\pi}\right)^n\prod_{k=1}^{k-n}\left(1-\frac{1}{4k^2}\right)^{k-n}.\end{aligned}$$

Next, we study the large-n asymptotics of the formula we derived. We want to show that

$$\left(\frac{2}{\pi}\right)^n\prod_{k=1}^{k-n}\left(1-\frac{1}{4k^2}\right)^{k-n}\sim 2^{\frac{1}{3}}e^{-3\zeta'(-1)}\cdot(2n)^{-\frac{1}{4}},\text{ as } n\to\infty.$$

At this point it is convenient to express D_n as

$$\left(\frac{2}{\pi}\right)^n\prod_{k=0}^{n-1}(k!)^2\cdot\frac{2^k}{(2k-1)!!}\cdot\frac{2^k}{(2k+1)!!}. \tag{4.10}$$

We compute the product in parts. First, using the basic properties of Barnes G-function [16, Section 5.17], we have

$$\prod_{k=0}^{n-1} k! = G(n+1),$$

where G is the Barnes G-function. Then, we have

$$\prod_{k=0}^{n-1} (k!)^2 = G(n+1)^2.$$

Next, we use the following identity for the Gamma function [1, Formula 26.2.28]:

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}.$$

With the help of this formula, we write

$$\frac{2^k}{(2k-1)!!} = \frac{\sqrt{\pi}}{\Gamma\left(k+\frac{1}{2}\right)}$$

and

$$\frac{2^k}{(2k+1)!!} = \frac{1}{2}\cdot\frac{\sqrt{\pi}}{\Gamma\left(k+1+\frac{1}{2}\right)}.$$

The Barnes G-function satisfies the relation

$$G(z+1) = \Gamma(z)G(z).$$

Next, we utilize this identity to express the Gamma-function as $\Gamma(z) = \frac{G(z+1)}{G(z)}$. The product of Gamma-functions becomes a telescopic product

$$\prod_{k=0}^{n-1} \Gamma\left(k+\frac{1}{2}\right) = \prod_{k=0}^{n-1} \frac{G\left(k+1+\frac{1}{2}\right)}{G\left(k+\frac{1}{2}\right)} = \frac{G(n+\frac{1}{2})}{G\left(\frac{1}{2}\right)},$$

and similarly

$$\prod_{k=0}^{n-1} \Gamma\left(k+\frac{3}{2}\right) = \frac{G(n+\frac{3}{2})}{G\left(\frac{3}{2}\right)}.$$

Using the expressions above, we write

$$\prod_{k=0}^{n-1} \frac{2^n}{(2k-1)!!} = \prod_{k=0}^{n-1} \frac{\sqrt{\pi}}{\Gamma\left(k+\frac{1}{2}\right)} = \pi^{n/2}\frac{G\left(\frac{1}{2}\right)}{G(n+\frac{1}{2})}$$

and
$$\prod_{k=0}^{n-1} \frac{2^n}{(2k+1)!!} = \prod_{k=0}^{n-1} \frac{1}{2} \cdot \frac{\sqrt{\pi}}{\Gamma\left(k+\frac{3}{2}\right)} = \pi^{n/2} 2^{-n} \frac{G\left(\frac{3}{2}\right)}{G(n+\frac{3}{2})}.$$
The entire product in (4.10) can be expressed as
$$\begin{aligned} &\left(\frac{2}{\pi}\right)^n \cdot G(n+1)^2 \cdot \pi^{n/2} \frac{G\left(\frac{1}{2}\right)}{G(n+\frac{1}{2})} \cdot \pi^{n/2} 2^{-n} \frac{G\left(\frac{3}{2}\right)}{G(n+\frac{3}{2})} \\ =&G\left(1/2\right) G\left(3/2\right) \frac{G(n+1)^2}{G(n+\frac{1}{2})G(n+\frac{3}{2})}. \end{aligned}$$
The values of the constants are
$$\begin{aligned} G(1/2) &= A^{-3/2} \pi^{-1/4} e^{1/8} 2^{1/24}, \\ G(3/2) &= A^{-3/2} \pi^{1/4} e^{1/8} 2^{1/24}, \end{aligned}$$
where A is the Glaisher-Kinkelin constant [8, Section 2.15]. Plugging these in yields
$$D_n = \frac{2^{1/12} e^{1/4}}{A^3} \frac{G(n+1)^2}{G(n+\frac{1}{2})G(n+\frac{3}{2})}.$$
Let us consider the asymptotics of the function. The Barnes G-function has the following asymptotic expansion, as $z \to \infty$:
$$\log G(z+1) = \frac{1}{12} - \log A + \frac{1}{2} z \log 2\pi + \left(\frac{1}{2} z^2 - \frac{1}{12}\right) \log z - \frac{3}{4} z^2 + O(z^{-2}).$$
Using the expansion, we write
$$\begin{aligned} G(n+1)^2 &= \left(e^{\log G(n+1)}\right)^2 = e^{2\left(\frac{1}{12} - \log A + \frac{1}{2} n \log 2\pi + \left(\frac{1}{2} n^2 - \frac{1}{12}\right) \log n - \frac{3}{4} n^2\right) + O(n^{-2})}, \\ G\left(n+1 \pm \tfrac{1}{2}\right) &= e^{\frac{1}{12} - \log A + \frac{1}{2}\left(n \pm \frac{1}{2}\right) \log 2\pi + \left(\frac{1}{2}\left(n \pm \frac{1}{2}\right)^2 - \frac{1}{12}\right) \log\left(n \pm \frac{1}{2}\right) - \frac{3}{4}\left(n \pm \frac{1}{2}\right)^2 + O(n^{-2})}. \end{aligned}$$
Thus, we have $\frac{G(n+1)^2}{G(n+\frac{1}{2})G(n+\frac{3}{2})} = e^{g(n)}$, where
$$\begin{aligned} g(n) = &\left(n^2 - \tfrac{1}{6}\right) \log n + \left(-\tfrac{1}{2} n^2 + \tfrac{1}{2} n - \tfrac{1}{24}\right) \log\left(n - \tfrac{1}{2}\right) \\ &+ \left(-\tfrac{1}{2} n^2 - \tfrac{1}{2} n - \tfrac{1}{24}\right) \log\left(n + \tfrac{1}{2}\right) + O\left(n^{-2}\right). \end{aligned}$$
Since we have
$$\log\left(z + \frac{1}{2}\right) = \log\left(z\left(1 + \frac{1}{2z}\right)\right) = \log z + \log(1+x)$$

for $x = \frac{1}{2z}$, we can use the Taylor series for the function $\log(1+x)$ to write

$$\log\left(z \pm \frac{1}{2}\right) = \log z \pm \frac{1}{2z} - \frac{1}{8z^2} + O(z^{-3}).$$

This asymptotic expansion allows us to write

$$\begin{aligned} g(n) &= \left(n^2 - \tfrac{1}{6}\right)\log n + \left(-\tfrac{1}{2}n^2 + \tfrac{1}{2}n - \tfrac{1}{24}\right)\left(\log n - \tfrac{1}{2n} - \tfrac{1}{8n^2} + O\left(n^{-3}\right)\right) \\ &\quad + \left(-\tfrac{1}{2}n^2 - \tfrac{1}{2}n - \tfrac{1}{24}\right)\left(\log n + \tfrac{1}{2n} - \tfrac{1}{8n^2} + O\left(n^{-3}\right)\right) + O\left(n^{-2}\right) \\ &= -\tfrac{1}{4}\log n + O\left(n^{-1}\right). \end{aligned}$$

Therefore, we have

$$\frac{G(n+1)^2}{G(n+\frac{1}{2})G(n+\frac{3}{2})} = e^{-\frac{1}{4}\log n + O(n^{-1})} \sim n^{-1/4}.$$

Now we see that the asymptotic formula for the diagonal spin-correlations at the critical temperature is

$$D_n \sim \frac{2^{1/12}e^{1/4}}{A^3}n^{-1/4} = \frac{2^{1/3}e^{1/4}}{A^3}(2n)^{-1/4}.$$

By substituting $\zeta'(-1) = \frac{1}{12} - \log A$ [8, 2.15], where ζ is the Riemann zeta function, this can be written in the form

$$D_n \sim 2^{1/3}e^{3\zeta'(-1)}(2n)^{-1/4}.$$

□

Chapter 5

Orthogonal Polynomials on the Unit Circle

At a subcritical temperature $\beta > \beta_{\text{crit}}$, the spin correlations can be analyzed using the same basic idea as above, utilizing the properties of the Fourier transform in Theorem 3.2. However, this time we do not have an explicit representation for the orthogonal polynomials involved, as we did at the critical temperature. In this chapter, we will cover some tools from the the theory of orthogonal polynomials on the unit circle, and we will use complex analysis to study the asymptotic properties of these polynomials. The article [18] provides a good overview on the central properties of orthogonal polynomials and the tools we will use below.

5.1 Essentials on the Theory of the Orthogonal Polynomials

First, we define the orthogonal polynomials on the unit circle. The rest of this section covers some useful basic properties that the orthogonal polynomials have.

Definition 5.1. *Let the weight function w be a function whose restriction on $\mathbb{T}$ is a non-negative finite function with a support of infinitely many points. The unique monic orthogonal polynomial Φ_n of degree n is the degree n polynomial that satisfies the orthogonality condition*

$$\int_0^{2\pi} e^{-\mathbb{i}j\theta}\Phi_n\left(e^{i\theta}\right) w\left(e^{\mathbb{i}\theta}\right) \mathrm{d}\theta = 0$$

for all $j = 0, \cdots, n-1$, and has 1 as the leading coefficient.

The orthogonal polynomials depend on the weight function w and exist, if w meets the conditions in the definition [18].

It will be useful to define an inner product on the polynomials. That allows us to use a shorter notation to express the integrals related to orthogonality conditions.

Definition 5.2. *Let us define an inner product $\langle \cdot, \cdot \rangle$ such that for polynomials p and q and for a real-valued weight function w we have*

$$\langle p, q \rangle = \frac{1}{2\pi} \int_0^{2\pi} p(e^{\mathbb{i}\theta})\overline{q(e^{\mathbb{i}\theta})} w(e^{\mathbb{i}\theta}) \mathrm{d}\theta.$$

One way to construct these polynomials is to use the Gram–Schmidt process based on the inner product with respect to the weight function w, starting with the linearly independent set of monomials $\{1, e^{\mathbb{i}t}, \cdots, e^{\mathbb{i}nt}\}$, or by setting $z = e^{\mathbb{i}t}$, the set $\{1, z, \cdots, z^n\}$, as the basis. We start by defining $p_0(z) = 1$. Now, by applying the Gram–Schmidt process we obtain a sequence of polynomials $(p_j)_{j=0}^n$. The polynomial p_j clearly is orthogonal to all the monomials z^k, $k = 0, \cdots, j-1$. Thus the monic orthogonal polynomial is

$$\Phi_n = \frac{p_n}{\chi_n},$$

where χ_n is the leading coefficient of p_n.

Let us define an operator ^{*,n} by

$$p^{*,n}\left(e^{\mathbb{i}\theta}\right) = e^{\mathbb{i}n\theta}\overline{p(e^{\mathbb{i}\theta})}.$$

It is usually unnecessary to explicitly state n since it is often clear that the polynomials we are dealing with have a certain degree. When applied to polynomials of degree n, this operator has the effect of reversing the order of the coefficients. If we have

$$p_n(z) = \sum_{k=0}^{n} c_k z^k,$$

then we can give the following alternate formulas for the operation, which also work outside the unit circle:

$$p_n^*(z) = \sum_{k=0}^{n} \overline{c_k} z^{n-k} = z^n \overline{p_n(1/\overline{z})}. \tag{5.1}$$

Especially, since Φ_n is monic, we have $\Phi_n^*(0) = 1$.

The following proposition shows that the orthogonal polynomials are unique up to the normalization constant.

Proposition 5.1. *Let p be a polynomial with* $\deg(p) \leq n$. *If* $\langle p, z^j \rangle = 0$ *for all* $j = 0, \cdots, n-1$, *then we have* $p = c\Phi_n$ *for some* $c \in \mathbb{C}$.

Moreover, if $\langle p, z^j \rangle = 0$ *for all* $j = 1, \cdots, n$, *then* $p = c^*\Phi_n^*$ *for some* $c^* \in \mathbb{C}$.

Proof. Clearly, we have $\langle c\Phi_n, z^j \rangle = c\langle \Phi_n, z^j \rangle = 0$ for all $j = 0, \cdots, n-1$ by the definition of Φ_n. The vector space P_n containing the polynomials of degree at most n has the dimension $\dim(P_n) = n+1$. Let us define a linear mapping $L : P_n \to \mathbb{R}^n$ such that

$$L(Q_n) = \begin{pmatrix} \langle Q_n, 1 \rangle \\ \langle Q_n, z^1 \rangle \\ \vdots \\ \langle Q_n, z^{n-1} \rangle \end{pmatrix}.$$

The polynomial Q_n satisfies the desired orthogonality condition if and only if it belongs to the null space of L, that is $L(Q_n) = \mathbf{0}$. L_n is a surjection, which can be seen by defining matrix $M \in \mathbb{C}^{n \times n}$ such that

$$\begin{aligned} M &= \begin{pmatrix} L(\Phi_0) & L(\Phi_1) & \cdots & L(\Phi_{n-1}) \end{pmatrix} \\ &= \begin{pmatrix} \|\Phi_0\|^2 & 0 & 0 & & 0 \\ \# & \|\Phi_1\|^2 & 0 & \cdots & 0 \\ \# & \# & \|\Phi_2\|^2 & & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \# & \# & \# & \cdots & \|\Phi_{n-1}\|^2 \end{pmatrix}, \end{aligned}$$

where the entries denoted by # are some values zero or non-zero. Since Φ_j is orthogonal to all the monomials z^k, $k = 0, \cdots, j-1$, M is a lower-triangular matrix. Thus all its column vectors are linearly independent, which implies that we can find a member in P_n such that it maps to an arbitrary vector in $\mathbb{R}^n$. Thus the dimension of the image of L is $\mathrm{Ran}(L) = n$. The rank-nullity theorem states

$$\dim(P_n) = \mathrm{Ran}(L) + \dim(\mathrm{Ker}(L)).$$

Therefore, we have $\dim(\mathrm{Ker}(L)) = 1$, which means that the subspace containing polynomials satisfying the orthogonality condition is 1-dimensional. This implies $Q_n = c\Phi_n$.

For the second part, we have $\langle p(z), z^j \rangle = 0$ for all $j = 1, \cdots, n$. This is equivalent to having

$$\langle z^n \overline{p^*(1/\overline{z})}, z^j \rangle = \langle p^*(z), z^{n-j} \rangle = 0 \text{ for all } j = 1, \cdots, n.$$

Thus, $\langle p^*(z), z^j \rangle = 0$ for all $j = 0, \cdots, n-1$, which by the first part of the proposition means $p^* = c\Phi_n$, implying that $p = c^*\Phi_n^*$ for some $c^* \in \mathbb{C}$. $\square$

The following corollary is used in the proof of Lemma 4.6.

Corollary 5.1. *Let P_n be a degree n polynomial such that $\langle P_n, z^j\rangle = 0$ for all $j = 1, \cdots, n$ and $P_n(0) = 0$. Then, we have $P_n(z) = 0$ for all z.*

Proof. The orthogonality condition implies by Proposition 5.1 that we have $P_n = \alpha\Phi_n^*$ for some $\alpha \in \mathbb{C}$. Since $\Phi_n^*(0) = 1$ and $P_n(0) = 0$, α has to be 0 implying $P_n(z) = 0$. □

The following lemma essentially shows that inner products of the form $\langle p, z^j\rangle$ are real-valued for a symmetric real weight.

Lemma 5.1. *Let p be a real polynomial and w a real weight function that satisfies $w(e^{i\theta}) = w(e^{-\mathrm{i}\theta})$. Then the integral*

$$\int_0^{2\pi} p(e^{\mathrm{i}\theta})e^{-\mathrm{i}j\theta}w(e^{\mathrm{i}\theta})\mathrm{d}\theta$$

is real-valued.

Proof. Let us denote $p(z) = \sum_{k=0}^n c_k z^k$. The integral can be written as

$$\int_0^{2\pi} \sum_{k=0}^n c_k e^{\mathrm{i}k\theta}e^{-\mathrm{i}j\theta}w(e^{\mathrm{i}\theta})\mathrm{d}\theta = \sum_{k=0}^n c_k \int_0^{2\pi} e^{\mathrm{i}(k-j)\theta}w(e^{\mathrm{i}\theta})\mathrm{d}\theta.$$

The imaginary part of the integral is

$$\begin{aligned}
\mathrm{Im}\int_0^{2\pi} e^{\mathrm{i}(k-j)\theta}w(e^{\mathrm{i}\theta})\mathrm{d}\theta &= \int_0^{2\pi} \mathrm{Im}\left(e^{\mathrm{i}(k-j)\theta}\right) w(e^{\mathrm{i}\theta})\mathrm{d}\theta \\
&= \int_0^{2\pi} \sin\left((k-j)\theta\right) w(e^{\mathrm{i}\theta})\mathrm{d}\theta \\
&= \int_{-\pi}^{\pi} \sin\left((k-j)\theta\right) w(e^{\mathrm{i}\theta})\mathrm{d}\theta
\end{aligned}$$

Now sin is an odd function and w is even, so their product is an odd function. Integrating an odd function over a symmetric interval yields 0. □

It is useful to know that the orthogonal polynomials we are discussing have real coefficients, as the following lemma shows.

Lemma 5.2. *Let w be a weight function that satisfies the conditions in Definition 5.1, and is symmetric such that $w(e^{\mathrm{i}t}) = w(e^{-\mathrm{i}t})$. All the coefficients of Φ_n are real.*

Proof. Let us observe the Gram–Schmidt process that we used to construct Φ_n. We started using the set $\{1, z, \cdots, z^n\}$ as the base. First, we have $p_0 = 1$. The following polynomials are obtained by

$$p_k(z) = z^k - \sum_{j=1}^{k-1} \frac{\langle p_j(z), z^k \rangle}{\langle p_j(z), p_j(z) \rangle} p_j(z).$$

Whether p_k is real depends on the inner products $\langle p_j(z), z^k \rangle$. Lemma 5.1 states that if p_j is a real polynomial and if the weight function is symmetric, then the inner product $\langle p_j(z), z^k \rangle$ is real. Clearly p_k has real coefficients if all the polynomials $(p_j)_{j=0}^{n-1}$ are real. Since $p_0 = 1$, it follows through induction that p_k is real for all k. □

Next we analyze a polynomial that is like an orthogonal polynomial, but it is not orthogonal to the monomial z^0. It turns out that it can be expressed as a linear combination of Φ_n and Φ_n^*.

Proposition 5.2. *Let Q_n be a polynomial that satisfies the condition*

$$\langle Q_n, z^j \rangle = 0 \textit{ for all } j = 1, \cdots n-1. \tag{5.2}$$

This implies that Q_n can be expressed as

$$Q_n = c_n \Phi_n + c_n^* \Phi_n^*,$$

for some constants $c_n, c_n^* \in \mathbb{C}$.

Proof. The proof will follow the same idea as the proof of Proposition 5.1. First, we will show that the claim holds in reversed direction.

$$\langle Q_n, z^j \rangle = \langle c_n \Phi_n + c_n^* \Phi_n^*, z^j \rangle = c_n \langle \Phi_n, z^j \rangle + c_n^* \langle \Phi_n^*, z^j \rangle$$

According to the definition of Φ_n we have $\langle \Phi_n, z^j \rangle = 0$ for all $j = 0, \cdots, n-1$. Also, $\langle \Phi_n^*, z^j \rangle = 0$ for all $j = 1, \cdots, n$. Therefore the sum equals zero for all $j = 1, \cdots, n-1$. Since Φ_n and Φ_n^* are linearly independent,

$$S = \{c_n \Phi_n + c_n^* \Phi_n^* \mid c_n, c_n^* \in \mathbb{C}\} \subset P_n$$

defines a 2-dimensional subspace in P_n that satisfies the condition (5.2). Next we will prove that there are no polynomials satisfying the condition not belonging to this particular subspace. Let us define a linear mapping $L : P_n \to \mathbb{C}^{n-1}$ such that

$$L(Q_n) = \begin{pmatrix} \langle Q_n, z \rangle \\ \langle Q_n, z^2 \rangle \\ \vdots \\ \langle Q_n, z^{n-1} \rangle \end{pmatrix}.$$

Now we can see that Q_n satisfies condition (5.2) if and only if $L(Q_n) = \mathbf{0}$, i.e. $Q_n \in \mathrm{Ker}(L)$.

Now we will show that L is a surjection. Let us consider the following matrix $M \in \mathbb{C}^{(n-1)\times(n-1)}$:

$$\begin{aligned} M &= \begin{pmatrix} L(\Phi_1) & L(\Phi_2) & \cdots & L(\Phi_{n-1}) \end{pmatrix} \\ &= \begin{pmatrix} \|\Phi_1\|^2 & 0 & 0 & & 0 \\ \# & \|\Phi_2\|^2 & 0 & \cdots & 0 \\ \# & \# & \|\Phi_3\|^2 & & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \# & \# & \# & \cdots & \|\Phi_{n-1}\|^2 \end{pmatrix} \end{aligned}$$

The matrix is lower-triangular, and therefore all the column vectors are linearly independent. This means that we can find a member in P_n such that it maps to an arbitrary vector in $\mathbb{C}^{n-1}$. This implies that $\mathrm{Ran}(L) = n - 1$.

The rank-nullity theorem states that

$$\dim(P_n) = \mathrm{Ran}(L) + \dim(\mathrm{Ker}(L)).$$

Now since $\dim(P_n) = n+1$ and $\mathrm{Ran}(L) = n-1$, we compute $\dim(\mathrm{Ker}(L)) = 2$. This is the dimension of S, so we can conclude that all the polynomials satisfying the condition (5.2) must belong to the subspace S. $\square$

It is important to note that c^* is not necessarily the complex conjugate of c.

Let us consider the polynomial $g = \Phi_{n+1} - z\Phi_n$ of degree n. Now, we have

$$\langle g, z^j \rangle = \langle \Phi_{n+1}, z^j \rangle - \langle z\Phi_n, z^j \rangle.$$

Clearly $\langle \Phi_{n+1}, z^j \rangle = 0$ holds for all $j = 0, \cdots, n$. Also, we have

$$\langle z\Phi_n, z^j \rangle = \langle \Phi_n, z^{j-1} \rangle = 0, \; j = 1, \cdots, n,$$

so we can see that $\langle \Phi_{n+1} - z\Phi_n, z^j \rangle = 0$ holds for all $j = 1, \cdots, n$. This implies, according to Proposition 5.1, that we have $\Phi_{n+1} - z\Phi_n = c\Phi_n^*$. Using the notation $c = -\overline{\alpha}_n$ to follow a common convention, we obtain the relation

$$\Phi_{n+1}(z) = z\Phi_n(z) - \overline{\alpha}_n \Phi_n^*(z). \tag{5.3}$$

This relation is known as the Szegő recursion. By assigning $z = 0$ and remembering that $\Phi_n^*(0) = 1$ (since Φ_n is monic), we obtain

$$\alpha_n = -\overline{\Phi_{n+1}(0)}.$$

5.2 Identities for the Spin Correlations

The relations we have discussed can be used to find alternative expressions for the diagonal spin correlations D_n, D_n^*, D_{n+1}, and D_{n+1}^*.

The following lemma is a technical result concerning the Fourier coefficients of a smooth function. It will be needed later in this section to derive the representations for the spin correlations. The result will be useful also later when proving that series involving Fourier coefficients of smooth functions converge. It states that the absolute values $\widehat{f}(k)$ of the Fourier coefficients of a smooth function f decrease at a rate faster than any polynomial, when the absolute value of k increases.

We denote the unit circle by $\mathbb{T}$.

Lemma 5.3. *Let $f \in C^\infty(\mathbb{T})$. Then, we have $\widehat{f}(k) = O(|k|^{-N})$ for any $N < \infty$.*

Proof. Integration by parts yields

$$\begin{aligned}
\widehat{f}(k) &= \int_0^{2\pi} e^{-\mathbb{i}k\theta} f(e^{\mathbb{i}\theta}) \frac{\mathrm{d}\theta}{2\pi} \\
&= -\frac{1}{2\pi}\Big|_0^{2\pi} e^{-\mathbb{i}k\theta} \frac{\mathrm{d}}{\mathrm{d}\theta} f(e^{\mathbb{i}\theta}) - \int_0^{2\pi} \frac{-1}{\mathbb{i}k} e^{-\mathbb{i}k\theta} \left(\frac{\mathrm{d}}{\mathrm{d}\theta} f(e^{\mathbb{i}\theta}) \right) \frac{\mathrm{d}\theta}{2\pi} \\
&= \int_0^{2\pi} \frac{1}{\mathbb{i}k} e^{-\mathbb{i}k\theta} \left(\frac{\mathrm{d}}{\mathrm{d}\theta} f(e^{\mathbb{i}\theta}) \right) \frac{\mathrm{d}\theta}{2\pi}.
\end{aligned}$$

After integrating by parts N times, we have

$$\widehat{f}(k) = \int_0^{2\pi} \frac{1}{(\mathbb{i}k)^N} e^{-\mathbb{i}k\theta} \left(\frac{\mathrm{d}^N}{\mathrm{d}\theta^N} f(e^{\mathbb{i}\theta}) \right) \frac{\mathrm{d}\theta}{2\pi}.$$

The absolute value of the above expression is

$$\begin{aligned}
|\widehat{f}(k)| &= \left| \int_0^{2\pi} \frac{1}{(\mathbb{i}k)^N} e^{-\mathbb{i}k\theta} \left(\frac{\mathrm{d}^N}{\mathrm{d}\theta^N} f(e^{\mathbb{i}\theta}) \right) \frac{\mathrm{d}\theta}{2\pi} \right| \\
&\le \int_0^{2\pi} \left| \frac{1}{(\mathbb{i}k)^N} e^{-\mathbb{i}k\theta} \frac{\mathrm{d}^N}{\mathrm{d}\theta^N} f(e^{\mathbb{i}\theta}) \right| \frac{\mathrm{d}\theta}{2\pi} \\
&\le |k|^{-N} \int_0^{2\pi} \left| e^{-\mathbb{i}k\theta} \right| \left| \frac{\mathrm{d}^N}{\mathrm{d}\theta^N} f(e^{\mathbb{i}\theta}) \right| \frac{\mathrm{d}\theta}{2\pi}.
\end{aligned}$$

The assumption $f \in C^\infty(\mathbb{T})$ implies that $\left| \frac{\mathrm{d}^N}{\mathrm{d}\theta^N} f(e^{\mathbb{i}\theta}) \right|$ is bounded for all N, θ, and thus, we have

$$\begin{aligned}
|\widehat{f}(k)| &\le |k|^{-N} \int_0^{2\pi} \sup_{\theta\in[0,2\pi]} \left| \frac{\mathrm{d}^N}{\mathrm{d}\theta^N} f(e^{\mathbb{i}\theta}) \right| \frac{\mathrm{d}\theta}{2\pi} \\
&= |k|^{-N} \sup_{\theta\in[0,2\pi]} \left| \frac{\mathrm{d}^N}{\mathrm{d}\theta^N} f(e^{\mathbb{i}\theta}) \right| \\
&= O\left(|k|^{-N}\right)
\end{aligned}$$

for any $N < \infty$. □

5.3 Back to the Ising model

Now we have covered some general theory concerning orthogonal polynomials on the unit circle. Next, we will apply the theory to the polynomial Q_n related to the Ising model.

Let us recall Theorem 3.2 and that we have

$$Q_n(e^{\mathbb{i}\theta})w(e^{\mathbb{i}\theta}) = D^*_{n+1} + q^2 D_{n+1} e^{\mathbb{i}n\theta} + \sum_{\substack{j<0\\ j>n}} \kappa_j e^{\mathbb{i}j\theta}, \tag{5.4}$$

where Q_n is a trigonometric polynomial of the form

$$Q_n(e^{\mathbb{i}\theta}) = D^*_n + \cdots D_n e^{\mathbb{i}n\theta}. \tag{5.5}$$

and the weight function is $w\left(e^{\mathbb{i}t}\right) = (1+q^2)\left(1-\left(m\cos\frac{t}{2}\right)^2\right)^{\frac{1}{2}}$. According to Proposition 5.2, Q_n can also be expressed as

$$Q_n(e^{\mathbb{i}\theta}) = c_n \Phi_n(e^{\mathbb{i}\theta}) + c^*_n \Phi^*_n(e^{\mathbb{i}\theta}),$$

where Φ_n is a monic orthogonal polynomial and c_n, c^*_n are constants in $\mathbb{C}$. We also recall Szegő recursion (5.3) which states $\Phi_{n+1}(z) = z\Phi_n(z) - \overline{\alpha}_n \Phi^*_n(z)$. The following proposition uses these and provides alternative expressions for the diagonal spin correlations D_n, D^*_n, D_{n+1}, and D^*_{n+1}.

Proposition 5.3. *The following identities hold:*

$$\begin{aligned}
D^*_n &= c^*_n - \alpha_{n-1} c_n, && (5.6)\\
D_n &= c_n - \alpha_{n-1} c^*_n, && (5.7)\\
D^*_{n+1} &= c^*_n \|\Phi^*_n\|^2, && (5.8)\\
q^2 D_{n+1} &= c_n \|\Phi_n\|^2. && (5.9)
\end{aligned}$$

Proof. To prove (5.8) and (5.9), we write (5.4) as

$$\left(c_n\Phi_n(e^{\mathrm{i}\theta}) + c_n^*\Phi_n^*(e^{\mathrm{i}\theta})\right) w(e^{\mathrm{i}\theta}) = D_{n+1}^* + q^2 D_{n+1} e^{\mathrm{i}n\theta} + \sum_{j\in\mathbb{Z}\setminus\{0,\dots,n\}} \kappa_j e^{\mathrm{i}j\theta}, \tag{5.10}$$

where κ_j are some constants. Next we integrate both sides of the equation over the unit circle, which yields

$$\begin{aligned}&\int_0^{2\pi} c_n\Phi_n(e^{\mathrm{i}\theta})w(e^{\mathrm{i}\theta})\frac{\mathrm{d}\theta}{2\pi} + \int_0^{2\pi} c_n^*\Phi_n^*(e^{\mathrm{i}\theta})w(e^{\mathrm{i}\theta})\frac{\mathrm{d}\theta}{2\pi}\\ =&D_{n+1}^*\int_0^{2\pi}\frac{\mathrm{d}\theta}{2\pi} + q^2 D_{n+1}\int_0^{2\pi} e^{\mathrm{i}n\theta}\frac{\mathrm{d}\theta}{2\pi} + \int_0^{2\pi}\left(\sum_{j\in\mathbb{Z}\setminus\{0,\dots,n\}}\kappa_j e^{\mathrm{i}j\theta}\right)\frac{\mathrm{d}\theta}{2\pi}.\end{aligned}$$

Since $\Phi_n \perp 1$ and $\langle\Phi_n^*, 1\rangle = \|\Phi_n^*\|^2$, the left-hand side simplifies to $c_n^*\|\Phi_n^*\|^2$. On the right-hand side the only non-zero term is D_{n+1}^*, because we have $\int_0^{2\pi} e^{\mathrm{i}j\theta}\mathrm{d}\theta = 0$ for all $j \neq 0$. We were able to use the dominated convergence theorem to interchange the integration and summation in the last term since the Fourier coefficients κ_j decay quickly by Lemma 5.3, which is because the weight function is smooth.

To prove (5.9), we multiply both sides of (5.10) by $e^{-\mathrm{i}n\theta}$, and then we integrate both sides. We also note that $\Phi_n^*(e^{\mathrm{i}\theta}) = e^{\mathrm{i}n\theta}\overline{\Phi_n(e^{\mathrm{i}\theta})}$, so we have

$$\begin{aligned}&\int_0^{2\pi} c_n e^{-\mathrm{i}n\theta}\Phi_n(e^{\mathrm{i}\theta})w(e^{\mathrm{i}\theta})\frac{\mathrm{d}\theta}{2\pi} + \int_0^{2\pi} c_n^* e^{-\mathrm{i}n\theta}e^{\mathrm{i}n\theta}\overline{\Phi_n^*(e^{\mathrm{i}\theta})}w(e^{\mathrm{i}\theta})\frac{\mathrm{d}\theta}{2\pi}\\ =&D_{n+1}^*\int_0^{2\pi} e^{-\mathrm{i}n\theta}\frac{\mathrm{d}\theta}{2\pi} + q^2 D_{n+1}\int_0^{2\pi}\frac{\mathrm{d}\theta}{2\pi} + \int_0^{2\pi}\left(\sum_{j\in\mathbb{Z}\setminus\{0,\dots,n\}}\kappa_j e^{\mathrm{i}(n-j)\theta}\right)\frac{\mathrm{d}\theta}{2\pi}.\end{aligned}$$

The left-hand side can be written as $c_n\langle\Phi_n, z^n\rangle + c_n^*\langle 1, \Phi_n\rangle = c_n\|\Phi_n\|^2 + 0$. The right-hand side yields $q^2 D_{n+1}$.

Next, we prove (5.6) and (5.7). Using (5.5) and Proposition 5.2, we obtain

$$Q_n(0) = D_n^* = c_n\Phi_n(0) + c_n^*\Phi_n^*(0).$$

Applying the Szegő recursion (5.3) yields $\Phi_n(0) = -\overline{\alpha}_{n-1}$ and $\Phi_n^*(0) = 1$. Therefore, we have

$$D_n^* = c_n^* - \overline{\alpha}_{n-1}c_n.$$

The number α is real, so we have $\overline{\alpha} = \alpha$. Also, c_n and c_n^* are real, which can be seen from (5.8) and (5.9). By applying the operation * on both sides of (5.5), we obtain

$$Q_n^*(e^{\mathrm{i}n\theta}) = D_n + \cdots + D_n^* e^{\mathrm{i}n\theta}.$$

Since $c_n, c_n^* \in \mathbb{R}$, we have $Q_n^*(e^{\mathrm{i}\theta}) = c_n \Phi_n^*(e^{\mathrm{i}\theta}) + c_n^* \Phi(e^{\mathrm{i}\theta})$. We conclude

$$Q_n^*(0) = D_n = c_n \Phi_n^*(0) + c_n^* \Phi_n(0) = c_n - c_n^* \overline{\alpha}_{n-1}.$$

□

With the help of the identities in the previous proposition, we can derive an identity that will later be central when determining the spin correlations.

Proposition 5.4. *The following identity holds:*

$$D_{n+1}^* \Phi_n^*(q^2) + q^2 D_{n+1} \Phi_n(q^2) = \prod_{j=0}^{n} \|\Phi_j\|^2.$$

Proof. Let us prove this through induction.

As a consequence of Theorem 3.1, we have $Q_0(e^{\mathrm{i}t}) = D_0 = D_0^* = 1$. This together with Proposition 5.2 implies that we have $c_0 + c_0^* = 1$. Hence, we see from the last two identities in Proposition 5.3, that we have

$$D_1^* + q^2 D_1 = \|\Phi_0\|^2.$$

This is our base case for the induction.

Now, what remains is to show that

$$D_{n+1}^* \Phi_n^*(q^2) + q^2 D_{n+1} \Phi_n(q^2) = \|\Phi_n\|^2 \left[D_n^* \Phi_{n-1}^*(q^2) + q^2 D_n \Phi_{n-1}(q^2) \right]$$

holds for $n \geq 1$. According to Proposition 5.3, the left-hand side can be written as

$$c_n^* \|\Phi_n^*\|^2 \Phi_n^*(q^2) + c_n \|\Phi_n\|^2 \Phi_n(q^2),$$

where $\|\Phi_n^*\|^2 = \|\Phi_n\|^2$, so we can write this as

$$\|\Phi_n\|^2 \left[c_n \Phi_n(q^2) + c_n^* \Phi_n^*(q^2) \right].$$

Next, we apply the Szegő-recursion to both Φ_n and Φ_n^* such that

$$\begin{aligned} \Phi_n(z) &= z\Phi_{n-1}(z) - \overline{\alpha}_{n-1} \Phi_{n-1}^*(z), \\ \Phi_n^*(z) &= \Phi_{n-1}^*(z) - \alpha_{n-1} z \Phi_{n-1}(z), \end{aligned}$$

which yields

$$\begin{aligned} &\|\Phi_n\|^2 \left[c_n \left(q^2 \Phi_{n-1}(q^2) - \overline{\alpha}_{n-1} q^2 \Phi_{n-1}^*(q^2) \right) + c_n^* \left(\Phi_{n-1}^* - \alpha_{n-1} q^2 \Phi_{n-1}(q^2) \right) \right] \\ =&\|\Phi_n\|^2 \left[\Phi_{n-1} q^2 (c_n - c_n^* \alpha_{n-1}) + \Phi_{n-1}^* \left(c_n^* - c_n \alpha_{n-1} \right) \right], \end{aligned}$$

as we remember that $\alpha \in \mathbb{R}$.

Using the formulas for D_n^* and $q^2 D_n$ in Proposition 5.3, we obtain the relation

$$\|\Phi_n\|^2 \left[D_n^* \Phi_{n-1}^*(q^2) + q^2 D_n \Phi_{n-1}(q^2) \right].$$

Hence, through the induction, the original claim holds. □

Chapter 6

The Leading Coefficients through Riemann–Hilbert Problems

Theorem 5.4 showed that the diagonal spin correlations can be solved by computing the product $\prod_{j=0}^{n}\|\Phi_j\|^2 = \prod_{j=0}^{n}\chi_j^{-2}$, where the identity holds because we have $\langle\Phi_j,\Phi_j\rangle = \langle\chi_j^{-1}p_j,\chi_j^{-1}p_j\rangle = \chi_j^{-2}\langle p_j,p_j\rangle = \chi_j^{-2}$. The objective of this chapter is to determine the asymptotic behavior of this product for the large values of n. First we prove some properties of integral transforms. Then, we represent the product using a Riemann–Hilbert problem. Finally, we prove a result called Szegő's theorem, which provides the asymptotics we are interested in.

6.1 Cacuhy Transform

Let us consider the following function that is defined in $\mathbb{C}\setminus\mathbb{T}$ as

$$I_f(z) = \oint_{\mathbb{T}} \frac{f(\omega)}{\omega - z}\frac{\mathrm{d}\omega}{2\pi\mathbb{i}}, \tag{6.1}$$

where $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ is the unit circle and we have $f \in L^1(\mathbb{T})$. The integration is performed in a counter-clockwise manner. All the coming complex integrals are also oriented in the counter-clockwise manner, even though it is not explicitly stated later. The function is defined both inside and outside the unit circle but not on the unit circle. This integral operator is called the Cauchy transform of f.

The following lemma shows that this function is holomorphic in its domain.

Lemma 6.1. *The function I_f is holomorphic in $\mathbb{C}\setminus\mathbb{T}$ for all $f \in L^2(\mathbb{T})$.*

Proof. We will show that I_f has a complex derivative at every point in $\mathbb{C}\setminus\mathbb{T}$. If the derivative exists, it is the limit of the difference quotient

$$\frac{1}{h}\left(\oint_{\mathbb{T}}\frac{f(\omega)}{\omega-(z+h)}\frac{d\omega}{2\pi\mathbb{i}}-\oint_{\mathbb{T}}\frac{f(\omega)}{\omega-z}\frac{\mathrm{d}\omega}{2\pi\mathbb{i}}\right)$$

as $h\to 0$, $h\in\mathbb{C}$. The expression can be written as

$$\begin{aligned}&\frac{1}{h}\oint_{\mathbb{T}}\left(\frac{f(\omega)}{\omega-(z+h)}-\frac{f(\omega)}{\omega-z}\right)\frac{\mathrm{d}\omega}{2\pi\mathbb{i}}\\ =&\frac{1}{h}\oint_{\mathbb{T}}\frac{hf(\omega)}{(\omega-z-h)(\omega-z)}\frac{\mathrm{d}\omega}{2\pi\mathbb{i}}\\ =&\oint_{\mathbb{T}}\frac{f(\omega)}{(\omega-z-h)(\omega-z)}\frac{\mathrm{d}\omega}{2\pi\mathbb{i}}.\end{aligned}$$

Taking the limit $h\to 0$ yields

$$\lim_{h\to 0}\oint_{\mathbb{T}}\frac{f(\omega)}{(\omega-z-h)(\omega-z)}\frac{\mathrm{d}\omega}{2\pi\mathbb{i}}=\oint_{\mathbb{T}}\lim_{h\to 0}\frac{f(\omega)}{(\omega-z-h)(\omega-z)}\frac{\mathrm{d}\omega}{2\pi\mathbb{i}}.$$

This holds due to the dominated convergence theorem, which can be applied using the following argument. Let $r=||z|-1|$. Assuming $|h|<r$, we have

$$\left|\frac{f(\omega)}{(\omega-z-h)(\omega-z)}\right|=\frac{|f(\omega)|}{|\omega-z-h|\,|\omega-z|}<\frac{|f(\omega)|}{\left|\frac{r-1}{2}\right|\,|r-1|}=\frac{2}{(r-1)^2}\,|f(\omega)|$$

The expression $\frac{2}{(r-1)^2}$ only depends on z. Also, for the compact set $\mathbb{T}$, $f\in L^2(\mathbb{T})\subset L^1(\mathbb{T})$ holds. Thus, $\frac{2}{(r-1)^2}|f(\omega)|$ is an integrable function that dominates the integrand, and the use of the dominated convergence theorem is justified.

The limit is

$$\oint_{\mathbb{T}}\lim_{h\to 0}\frac{f(\omega)}{(\omega-z-h)(\omega-z)}\frac{\mathrm{d}\omega}{2\pi\mathbb{i}}=\oint_{\mathbb{T}}\frac{f(\omega)}{(\omega-z)^2}\frac{\mathrm{d}\omega}{2\pi\mathbb{i}},$$

which is the complex derivative of I_f. The limit exists for all $|z|\neq 1$ so I_f is holomorphic in $\mathbb{C}\setminus\mathbb{T}$. □

Let us define the following functions defined on the unit circle, $z\in\mathbb{T}$ and $\varepsilon>0$:

$$\begin{aligned}I_{f,+}^{(\varepsilon)}(z)&=\oint_{\mathbb{T}}\frac{f(\omega)}{\omega-(1-\varepsilon)z}\frac{\mathrm{d}\omega}{2\pi\mathbb{i}},\\ I_{f,-}^{(\varepsilon)}(z)&=\oint_{\mathbb{T}}\frac{f(\omega)}{\omega-(1+\varepsilon)z}\frac{\mathrm{d}\omega}{2\pi\mathbb{i}}.\end{aligned}$$

As we let $\varepsilon \searrow 0$, the limits for $I^{(\varepsilon)}_{f,+}(z)$ and $I^{(\varepsilon)}_{f,-}(z)$ will correspond to the limit of I_f when approaching the unit circle radially from from the inside or the outside, respectively.

These transforms affect the Fourier coefficients such that the first transform removes all the Fourier coefficients of negative order, and the second one removes the coefficients of orders $k \geq 0$, as shown in the next lemma.

Lemma 6.2. *Let $f \in L^2(\mathbb{T})$ and $\varepsilon > 0$. The functions $I^{(\varepsilon)}_{f,\pm}(z)$ can be expressed as*

$$I^{(\varepsilon)}_{f,+}(z) = \sum_{k=0}^{\infty} (1-\varepsilon)^k z^k \widehat{f}(k)$$

$$I^{(\varepsilon)}_{f,-}(z) = -\sum_{k=-\infty}^{-1} (1+\varepsilon)^k z^k \widehat{f}(k),$$

where $\widehat{f}(k)$ are the Fourier coefficients of f, $\widehat{f}(k) = \oint_{\mathbb{T}} \omega^k f(\omega) \frac{\mathrm{d}\omega}{2\pi \mathbb{i}\omega}$ and $z \in \mathbb{T}$.

Proof. The assumption $f \in L^2(\mathbb{T})$ implies that f is integrable on the unit circle, because for the compact set $\mathbb{T}$ we have $L^2(\mathbb{T}) \subset L^1(\mathbb{T})$.

Let us rewrite $I^{(\varepsilon)}_{f,+}(z)$ as

$$\oint_{\mathbb{T}} \frac{f(\omega)}{1-(1-\varepsilon)\frac{z}{\omega}} \frac{\mathrm{d}\omega}{2\pi \mathbb{i}\omega}.$$

Since $|(1-\varepsilon)\frac{z}{\omega}| \leq 1-\varepsilon$, we can use a geometric sum to write the integral as

$$\oint_{\mathbb{T}} f(\omega) \sum_{k=0}^{\infty} (1-\varepsilon)^k z^k \omega^{-k} \frac{\mathrm{d}\omega}{2\pi \mathbb{i}\omega}.$$

We notice, that the integrand satisfies

$$\begin{aligned} |f(\omega) \sum_{k=0}^{\infty} (1-\varepsilon)^k z^k \omega^{-k}| &\leq |f(\omega)| \sum_{k=0}^{\infty} (1-\varepsilon)^k \left|z^k \omega^{-k}\right| \\ &= |f(\omega)| \sum_{k=0}^{\infty} (1-\varepsilon)^k = \frac{1}{\varepsilon} |f(\omega)|, \end{aligned}$$

which is an integrable function multiplied by a constant. Therefore, we can use Fubini's theorem to interchange the order of summation and integration, so we can write $I^{(\varepsilon)}_{f,+}(z)$ as

$$\sum_{k=0}^{\infty} (1-\varepsilon)^k z^k \oint_{\mathbb{T}} \omega^{-k} f(\omega) \frac{\mathrm{d}\omega}{2\pi \mathbb{i}\omega}$$

Since we have $\widehat{f}(k) = \oint_{\mathbb{T}} \omega^{-k} f(\omega) \frac{\mathrm{d}\omega}{2\pi\mathbb{i}\omega}$, the above expression can be written as

$$\sum_{k=0}^{\infty} (1-\varepsilon)^k z^k \widehat{f}(k).$$

The corresponding expression for $I_{f,-}^{(\varepsilon)}$ is derived using the same idea. Since $|(1+\varepsilon)z| > 1$, we will write

$$I_{f,-}^{(\varepsilon)}(z) = \oint_{\mathbb{T}} \frac{f(\omega)}{\omega - (1+\varepsilon)z} \frac{\mathrm{d}\omega}{2\pi\mathbb{i}} = -\frac{1}{(1+\varepsilon)z} \oint_{\mathbb{T}} \frac{f(\omega)}{1 - \frac{\omega}{(1+\varepsilon)z}} \frac{\mathrm{d}\omega}{2\pi\mathbb{i}}.$$

This can be expanded as a geometric series ($|\frac{\omega}{(1+\varepsilon)z}| < 1$), which yields

$$I_{f,-}^{(\varepsilon)}(z) = -\frac{1}{(1+\varepsilon)z} \oint_{\mathbb{T}} \sum_{k=0}^{\infty} \omega^k (1+\varepsilon)^{-k} z^{-k} f(\omega) \frac{\mathrm{d}\omega}{2\pi\mathbb{i}}.$$

Rearranging some terms and using Fubini's theorem gives us the following expression:

$$\begin{aligned}
I_{f,-}^{(\varepsilon)}(z) &= -\sum_{k=0}^{\infty} (1+\varepsilon)^{-k-1} z^{-k-1} \oint_{\mathbb{T}} \omega^{k+1} f(\omega) \frac{\mathrm{d}\omega}{2\pi\mathbb{i}\omega} \\
&= -\sum_{k=0}^{\infty} (1+\varepsilon)^{-k-1} z^{-k-1} \widehat{f}(-k-1) \\
&= -\sum_{k=-\infty}^{-1} (1+\varepsilon)^k z^k \widehat{f}(k).
\end{aligned}$$

□

With the same reasoning as in the previous lemma, we have

$$I_f(z) = \begin{cases} \sum_{k=0}^{\infty} z^k \widehat{f}(k), & |z| < 1 \\ -\sum_{k=-\infty}^{-1} z^k \widehat{f}(k), & |z| > 1. \end{cases} \tag{6.2}$$

The following functions are of the same form as I_f, but this time on the unit cirlce. It will be shown later that these functions are the limits of I_f when approaching the unit circle from the inside and outside.

Definition 6.1. *Let $f \in L^2(\mathbb{T})$ and $z \in \mathbb{C}$. Let us define*

$$J_{f,+}(z) = \sum_{k=0}^{\infty} z^k \widehat{f}(k),$$

$$J_{f,-}(z) = -\sum_{k=-\infty}^{-1} z^k \widehat{f}(k).$$

Next, we will show that these functions are $L^2(\mathbb{T})$ functions, as well. Also, the L^2-norm of the function $J_{f,\pm}$ is less or equal to the L^2-norm of the function f.

Lemma 6.3. *Let $f \in L^2(\mathbb{T})$. Then, we have*

$$J_{f,\pm} \in L^2(\mathbb{T}).$$

Moreover, we have

$$\int_{\mathbb{T}} |J_{f,\pm}(\omega)|^2 \frac{\mathrm{d}\omega}{2\pi \mathbb{i}\omega} \leq \int_{\mathbb{T}} |f(\omega)|^2 \frac{\mathrm{d}\omega}{2\pi \mathbb{i}\omega}.$$

Proof. By the definition of Fourier series, we have $f(z) = \sum_{k=-\infty}^{\infty} z^k \widehat{f}(k)$. The monomials z^k form an orthonormal basis in $L^2(\mathbb{T})$ and we assumed $f \in L^2(\mathbb{T})$. Therefore, by the Parseval's formula for Hilbert spaces [20, Theorem 6.10] we have

$$\sum_{k=-\infty}^{\infty} \left|\widehat{f}(k)\right|^2 = \oint_{\mathbb{T}} |f(\omega)|^2 \frac{\mathrm{d}\omega}{2\pi \mathbb{i}\omega} < \infty.$$

Since we have $J_{f,+}(z) = \sum_{k=0}^{\infty} z^k \widehat{f}(k)$, we can write

$$\oint_{\mathbb{T}} |J_{f,+}(\omega)|^2 \frac{\mathrm{d}\omega}{2\pi \mathbb{i}\omega} = \sum_{k=-\infty}^{\infty} \left|\widehat{J_{f,+}}(k)\right|^2 = \sum_{k=0}^{\infty} \left|\widehat{f}(k)\right|^2 \leq \sum_{k=-\infty}^{\infty} \left|\widehat{f}(k)\right|^2 < \infty,$$

and similarly

$$\oint_{\mathbb{T}} |J_{f,-}(\omega)|^2 \frac{\mathrm{d}\omega}{2\pi \mathbb{i}\omega} = \sum_{k=-\infty}^{-1} \left|\widehat{f}(k)\right|^2 \leq \sum_{k=-\infty}^{\infty} \left|\widehat{f}(k)\right|^2 < \infty.$$

□

The following lemma shows that the Cauchy transform of a function converges to the function $J_{f,\pm}$ when the unit circle is approached from the inside or outside.

Lemma 6.4. *$I_{f,\pm}^{(\varepsilon)}$ converges to $J_{f,\pm}$ in $L^2(\mathbb{T})$, i.e. we have*

$$\lim_{\varepsilon \to 0^+} \left\| I_{f,\pm}^{(\varepsilon)} - J_{f,\pm} \right\|_{L^2(\mathbb{T})}^2 = 0.$$

Proof. Using the definitions of $I_{f,+}^{(\varepsilon)}$ and $J_{f,+}$, we can write

$$\left\| I_{f,+}^{(\varepsilon)} - J_{f,+} \right\|_{L^2(\mathbb{T})}^2 = \left\| \sum_{k=0}^{\infty} \left((1-\varepsilon)^k - 1 \right) z^k \widehat{f}(k) \right\|_{L^2(\mathbb{T})}^2 .$$

The Fourier-coefficients of the function inside the norm are $\left((1-\varepsilon)^k - 1\right) \widehat{f}(k)$ for $k \geq 0$ and 0 otherwise. Therefore applying the Parseval's formula yields

$$\left\| \sum_{k=0}^{\infty} \left((1-\varepsilon)^k - 1 \right) z^k \widehat{f}(k) \right\|_{L^2(\mathbb{T})}^2 = \sum_{k=0}^{\infty} \left((1-\varepsilon)^k - 1 \right)^2 \left| \widehat{f}(k) \right|^2 .$$

Since we have

$$\sum_{k=0}^{\infty} \left((1-\varepsilon)^k - 1 \right)^2 \left| \widehat{f}(k) \right|^2 \leq \sum_{k=0}^{\infty} \left| \widehat{f}(k) \right|^2 \leq \|f\|_{L^2(\mathbb{T})}^2 < \infty,$$

we can use the dominated convergence theorem to conclude that the limit is

$$\lim_{\varepsilon \to 0^+} \sum_{k=0}^{\infty} \left((1-\varepsilon)^k - 1 \right)^2 \left| \widehat{f}(k) \right|^2 = \sum_{k=0}^{\infty} \lim_{\varepsilon \to 0^+} \left((1-\varepsilon)^k - 1 \right)^2 \left| \widehat{f}(k) \right|^2 = 0.$$

The proof for $I_{f,-}^{(\varepsilon)}$ is essentially identical. □

From Definition 6.1 and the definition of the Fourier series, we obtain

$$J_{f,+}(z) - J_{f,-}(z) = f(z). \tag{6.3}$$

In the proofs, we always approached the unit circle radially when taking the limits. Actually, the results hold for non-radial approaches as long as z does not approach the integration path tangentially and if f is Hölder continuous [15, Chapter 2]. Later, the limits are always considered in this sense when approaching the unit circle. The functions we will use are Hölder continuous, so we will be able to use this result.

6.2 A Riemann–Hilbert problem

Next, we will define a Riemann–Hilbert problem. We define boundary conditions for a matrix valued function. It will turn out that the problem has a unique solution. The solution is a matrix valued function that contains information about the orthogonal polynomial of a certain degree. This function will play a critical role in the proof of Szegő's theorem later.

Lemma 6.5. *Let $e^V : \mathbb{C} \to \mathbb{R}$ be a smooth weight function and let $j \in \mathbb{N}$. The solution of the following problem is unique, assuming it exists.*

1. *$Y : \mathbb{C} \setminus \mathbb{T} \to \mathbb{C}^{2\times 2}$ is holomorphic*

2. *There exist functions $Y_\pm$ continuous on the unit circle such that*

$$\lim_{\varepsilon \to 0^+} Y\left((1 \mp \varepsilon)z\right) = Y_\pm(z), \; z \in \mathbb{T},$$

   which holds also when approaching the unit circle in any non-tangential way. The inside and outside limits satisfy

$$Y_+(z) = Y_-(z) \begin{pmatrix} 1 & z^{-j}e^{V(z)} \\ 0 & 1 \end{pmatrix}.$$

3. *We have*

$$Y(z) = \left(I + O(z^{-1})\right) \begin{pmatrix} z^j & 0 \\ 0 & z^{-j} \end{pmatrix},$$

   as $|z| \to \infty$.

Proof. Let Y be a solution that satisfies the three conditions above. The function is assumed holomorphic in $\mathbb{C} \setminus \mathbb{T}$, so also its determinant $\det(Y)$ is holomorphic in that domain. According to the condition 3, we have

$$\det\left(Y(z)\right) = \det \begin{pmatrix} z^j + O(z^{j-1}) & O(z^{-j-1}) \\ O(z^{j-1}) & z^{-j} + O(z^{-j-1}) \end{pmatrix} = 1 + O(z^{-1}).$$

We will show that $\det(Y)$ is holomorphic and bounded in the entire $\mathbb{C}$. We have

$$\det(Y_+(z)) = \det(Y_-(z)) \det \begin{pmatrix} 1 & z^{-j}e^{V(z)} \\ 0 & 1 \end{pmatrix} = \det(Y_-(z)),$$

which implies that $\det(Y(z))$ is continuous over $\mathbb{T}$. Let us consider a closed, piecewise smooth curve γ. We can split the area surrounded by γ in finitely many parts surrounded by paths γ_k such that each path is either entirely inside or outside the unit disc, or divided in two parts by the unit circle. Now, we have

$$\int_\gamma \det Y(z)\mathrm{d}z = \sum_k \int_{\gamma_k} \det Y(z)\mathrm{d}z.$$

If γ_k is entirely inside or outside the unit disc, by the holomorphicity of $\det Y(z)$ in $\mathbb{C} \setminus \mathbb{T}$ and Cauchy's integral theorem, the corresponding integral yields 0. If γ_k crosses the unit circle, then we split it in two loops, γ_k^+ inside the unit circle and γ_k^- outisde. The parts of γ_k^+ and γ_k^- that are along the unit circle are integrated using $\det Y_+(z)$ and $\det Y_-(z)$. Because these two functions are equal on $\mathbb{T}$ and the line is integrated in the opposite directions, these two integrals cancel each other yielding 0. Thus $\int_\gamma \det Y(z)\mathrm{d}z = 0$ and by Morera's theorem, $\det Y$ is holomorphic in $\mathbb{C}$.

Since $\det(Y)$ is holomorphic and bounded, Liouville's theorem implies that $\det(Y)$ is a constant function. Together with the fact $\det(Y(z)) = 1 + O(z^{-1})$ this implies $\det(Y(z)) = 1$ for all $z \in \mathbb{C}$.

The inverse matrix Y^{-1} exists because the determinant of Y is non-zero. This inverse matrix is

$$Y(z)^{-1} = \begin{pmatrix} Y_{22}(z) & -Y_{12}(z) \\ -Y_{21}(z) & Y_{11}(z) \end{pmatrix},$$

which is holomorphic in $\mathbb{C} \setminus \mathbb{T}$. We know that $Y_+ = Y_- J$ with

$$J = \begin{pmatrix} 1 & z^{-j}e^{V(z)} \\ 0 & 1 \end{pmatrix}.$$

Assume that $\widetilde{Y}$ is another solution to the problem. Both Y and $\widetilde{Y}^{-1}$ are holomorphic in $\mathbb{C} \setminus \mathbb{T}$, and so is $Y\widetilde{Y}^{-1}$. The function $Y\widetilde{Y}^{-1}$ is continuous over $\mathbb{T}$, because we have

$$\begin{aligned} Y_+(z)\widetilde{Y}_+(z)^{-1} &= Y_-(z)J(z)\left(\widetilde{Y}_-(z)J(z)\right)^{-1} \\ &= Y_-(z)J(z)J(z)^{-1}\widetilde{Y}_-(z)^{-1} \\ &= Y_-(z)\widetilde{Y}_-(z)^{-1}. \end{aligned}$$

Therefore, with the same kind of argument as earlier, $Y(z)\widetilde{Y}(z)^{-1}$ is holomorphic in $\mathbb{C}$.

As we let $|z| \to \infty$, we obtain

$$\begin{aligned} &Y(z)\widetilde{Y}(z)^{-1} \\ =&\begin{pmatrix} z^j + O(z^{j-1}) & O(z^{-j-1}) \\ O(z^{j-1}) & z^{-j} + O(z^{-j-1}) \end{pmatrix} \begin{pmatrix} z^{-j} + O(z^{-j-1}) & O(z^{-j-1}) \\ O(z^{j-1}) & z^j + O(z^{j-1}) \end{pmatrix} \\ =&I + O(z^{-1}). \end{aligned}$$

The function is bounded and holomorphic. Using Liouville's theorem, we conclude that

$$Y(z)\widetilde{Y}(z)^{-1} = I,$$

which implies $Y(z) = \widetilde{Y}(z)$. □

We just showed that if a certain Riemann–Hilbert problem has a solution, it is unique. The solution for this problem was first presented in [9]. In the next lemma, we show that this solution indeed solves the Riemann–Hilbert problem from the previous lemma.

Lemma 6.6. *Let $e^{V(z)}$ be a weight function that satisfies the conditions for a weight function in 5.1. Let p_j be the orthogonal polynomial of degree p_j with respect to the weight $e^{V(z)}$, and let its leading coefficient be χ_j. Assume, that p_j is normalized such that we have $\langle p_j, p_k\rangle = \delta_{j,k}$ and $\chi_j > 0$, implying that we have $p_j = \chi_j z^j + O(z^{j-1})$. Let p_j^* be as in (5.1). The unique solution for the Riemann–Hilbert problem presented in Lemma 6.5 is*

$$Y(z) = \begin{pmatrix} \dfrac{1}{\chi_j} p_j(z) & \dfrac{1}{\chi_j} \displaystyle\oint_{\mathbb{T}} \frac{p_j(\omega)}{\omega - z} \frac{e^{V(\omega)} \mathrm{d}\omega}{2\pi \mathbb{i} \omega^j} \\ -\chi_{j-1} p_{j-1}^*(z) & -\chi_{j-1} \displaystyle\oint_{\mathbb{T}} \frac{p_{j-1}^*(\omega)}{\omega - z} \frac{e^{V(\omega)} \mathrm{d}\omega}{2\pi \mathbb{i} \omega^j} \end{pmatrix}.$$

Proof. Each entry in the matrix is holomorphic in $\mathbb{C} \setminus \mathbb{T}$ and therefore Y is holomorphic as well.

Next we need to show that the limits $Y_\pm(z) = \lim_{\varepsilon\to 0^+} Y((1 \mp \varepsilon)z)$ exist when $z \in \mathbb{T}$. This is trivial for Y_{11} and Y_{21}, which are polynomials and thus continuous in $\mathbb{C}$. Let us define functions

$$f(z) = \frac{1}{\chi_j} \frac{p_j(z) e^{V(z)}}{z^j}, \qquad g(z) = -\chi_{j-1} \frac{p_{j-1}^*(z) e^{V(z)}}{z^j}.$$

Now, we can write

$$\begin{aligned} Y_{12}(z) &= \oint_{\mathbb{T}} \frac{f(\omega)}{\omega - z} \frac{\mathrm{d}\omega}{2\pi \mathbb{i}} = I_f(z), \\ Y_{22}(z) &= \oint_{\mathbb{T}} \frac{g(\omega)}{\omega - z} \frac{\mathrm{d}\omega}{2\pi \mathbb{i}} = I_g(z). \end{aligned}$$

Using (6.2), we conclude that $Y_{12,\pm} = J_{f,\pm}$. Therefore, the limits $Y_\pm$ exist.

Next, we need to show that

$$Y_+(z) = Y_-(z) \begin{pmatrix} 1 & z^{-j} e^{V(z)} \\ 0 & z^{-j} \end{pmatrix}. \tag{6.4}$$

The right-hand side of the identity can be written as

$$\begin{pmatrix} Y_{11,-}(z) & Y_{12,-}(z) \\ Y_{21,-}(z) & Y_{22,-}(z) \end{pmatrix} \begin{pmatrix} 1 & z^{-j} e^{V(z)} \\ 0 & z^{-j} \end{pmatrix} = \begin{pmatrix} Y_{11,-}(z) & Y_{11,-}(z) z^{-j} e^{V(z)} + Y_{12,-}(z) \\ Y_{21,-}(z) & Y_{21,-}(z) z^{-j} e^{V(z)} + Y_{22,-}(z) \end{pmatrix}.$$

Now we will go through each element in the matrix and show that they match the elements in Y_+. We have $Y_{11,-}(z) = Y_{11,+}(z)$ and $Y_{11,-}(z) = Y_{11,+}(z)$ because Y_{11} and Y_{22} are polynomials, which being continuous have unique limits that do not depend on how the unit circle is approached. Next, we evaluate the remaining elements of the matrix. We have

$$\begin{aligned}
Y_{11,-}(z)z^{-j}e^{V(z)} + Y_{12,-}(z) &= \frac{1}{\chi_j}p_j(z)z^{-j}e^{V(z)} + Y_{12,-}(z) \\
&= f(z) + Y_{12,-}(z) \\
&= J_{f,+}(z) - J_{f,-}(z) + J_{f,-}(z) \\
&= J_{f,+}(z).
\end{aligned}$$

We used the formulas (6.3) and $Y_{12,-}(z) = J_{f,-}(z)$. The last element in the matrix is computed in a similar way as

$$\begin{aligned}
Y_{21,-}(z)z^{-j}e^{V(z)} + Y_{22,-}(z) &= -\chi_j p^*_{j-1}(z)z^{-j}e^{V(z)} + Y_{22,-}(z) \\
&= g(z) + Y_{12,-}(z) \\
&= J_{g,+}(z) - J_{g,-}(z) + J_{g,-}(z) \\
&= J_{g,+}(z).
\end{aligned}$$

Therefore, (6.4) holds.

Let us examine the asymptotic behavior of Y as $|z| \to \infty$. Clearly, we have

$$Y_{11}(z) = \frac{1}{\chi_j}p_j(z) = z^j + O(z^{j-1}). \tag{6.5}$$

and

$$Y_{21}(z) = -\chi_{j-1}p^*_{j-1}(z) = O(z^{j-1}). \tag{6.6}$$

In order to study $Y_{12}(z)$, we apply a geometric series expansion, which yields

$$\begin{aligned}
Y_{12}(z) &= \frac{1}{\chi_j}\oint_{\mathbb{T}} \frac{p_j(\omega)}{\omega - z}\frac{e^{V(\omega)}\mathrm{d}\omega}{2\pi\mathbb{i}\omega^j} = -\frac{1}{\chi_j}\oint_{\mathbb{T}} \frac{1}{z}\frac{p_j(\omega)}{1-\frac{\omega}{z}}\frac{e^{V(\omega)}\mathrm{d}\omega}{2\pi\mathbb{i}\omega^j} \\
&= -\frac{1}{\chi_j}\oint_{\mathbb{T}}\sum_{k=0}^{\infty}\omega^{k-j+1}z^{-k-1}p_j(\omega)\frac{e^{V(\omega)}\mathrm{d}\omega}{2\pi\mathbb{i}\omega}
\end{aligned}$$

Now, we use the dominated convergence theorem to change the order of the integration and summation. The argument is similar to the earlier ones. Therefore we can write

$$\begin{aligned}
Y_{12}(z) &= -\frac{1}{\chi_j}\sum_{k=0}^{\infty}z^{-k-1}\oint_{\mathbb{T}}\omega^{k-j+1}p_j(\omega)\frac{e^{V(\omega)}\mathrm{d}\omega}{2\pi\mathbb{i}\omega} \\
&= -\frac{1}{\chi_j}\sum_{k=0}^{\infty}z^{-k-1}\langle p_j(\omega), \omega^{j-k-1}\rangle.
\end{aligned}$$

Since $\langle p_j(\omega), \omega^{j-k-1}\rangle = 0$ for all $k = 0, \dots, j-1$, we have

$$Y_{12}(z) = -\frac{1}{\chi_j} \sum_{k=j}^{\infty} z^{-k-1} \langle p_j(\omega), \omega^{j-k-1}\rangle = O\left(z^{-j-1}\right). \tag{6.7}$$

This is because p and the weight function are smooth, so by Lemma 5.3, the Fourier coefficients diminish quickly.

Similarly, we have

$$\begin{aligned}
Y_{22}(z) &= -\chi_{j-1} \oint_{\mathbb{T}} \frac{p^*_{j-1}(\omega)}{\omega - z} \frac{e^{V(\omega)} \mathrm{d}\omega}{2\pi \mathbb{i} \omega^j} \\
&= \chi_{j-1} \sum_{k=0}^{\infty} z^{-k-1} \oint_{\mathbb{T}} p^*_{j-1}(\omega) \omega^{k-j+1} \frac{e^{V(\omega)} \mathrm{d}\omega}{2\pi \mathbb{i} \omega} \\
&= \chi_{j-1} \sum_{k=0}^{\infty} z^{-k-1} \oint_{\mathbb{T}} \omega^{j-1} \overline{p_{j-1}(\omega)} \omega^{k-j+1} \frac{e^{V(\omega)} \mathrm{d}\omega}{2\pi \mathbb{i} \omega} \\
&= \chi_{j-1} \sum_{k=0}^{\infty} z^{-k-1} \langle \omega^k, p_{j-1}\rangle.
\end{aligned}$$

The inner product above satisfies $\langle \omega^k, p_{j-1}\rangle = \overline{\langle p_{j-1}(\omega), \omega^k\rangle} = 0$ for all $k = 0, \cdots, j-2$. For the case $k = j-1$, we recall that χ_{j-1} is real, and obtain

$$\overline{\langle p_{j-1}(\omega), \omega^{j-1}\rangle} = \frac{1}{\chi_{j-1}} \overline{\langle p_{j-1}(\omega), \chi_{j-1}\omega^{j-1}\rangle} = \frac{1}{\chi_{j-1}} \left\| p_{j-1} \right\|^2 = \frac{1}{\chi_{j-1}}.$$

Hence, we have

$$Y_{22}(z) = \chi_{j-1} z^{-j} \frac{1}{\chi_{j-1}} + \chi_{j-1} \sum_{k=j}^{\infty} z^{-k-1} \langle \omega^{k+1}, p_{j-1}\rangle = z^{-j} + O(z^{-j-1}).$$

□

6.3 The Product of the Leading Coefficients

Let us go back to the problem of computing the product $\prod_{j=0}^{N-1} \chi_j^{-2}$ for a large N. The following definition provides some functions we will need later. The idea is to modify the weight function so that we can get the convex combinations between the original weight function and the constant 1 by changing a parameter value. This will allow e.g. differentiation with respect to this parameter, which will be useful later.

Definition 6.2. *Let $t \in [0,1]$. Then $p_j(z;t)$ is defined as the orthonormal polynomial with respect to the weight function e^{tV}. Let $\chi_j(t) > 0$ be the leading coefficient of $p_j(z;t)$.*

Using this definition, $\chi_j(1) = \chi_j$ is the leading coefficient of $p_j(z) = p(j;1)$, and $\chi_j(0)$ is the leading coefficient of the orthogonal polynomial with respect to the constant measure $w(z) = 1$ on the unit circle. With this measure, the function z^j is orthogonal to all the monomials z^k with $k \neq j$, so we have $p_j(z;0) = z^j$ and $\chi_j(0) = 1$. Since $\chi_j(0) = 1$ for all j, we can write

$$\log \prod_{j=0}^{N-1} \chi_j(1)^{-2} = \int_0^1 \partial_t \log \prod_{j=0}^{N-1} \chi_j(t)^{-2} \mathrm{d}t. \tag{6.8}$$

Let us make an argument for the existence of the derivative in (6.8). We assumed that e^V is a smooth and real-valued function. Then e^{tV} is smooth and real-valued, as well, and the orthogonal polynomials exist for all $t \in [0,1]$, which implies that $\chi_j(t) > 0$ for all j and t. Let us recall how we constructed the orthogonal polynomials using Gram–Schmidt process. In this construction, values of $\chi_j(t)$ for different j are moments of e^{tV}, and therefore $\chi_j(t)$ is smooth and thus differentiable for all j.

The objective is to examine the asymptotics of the right-hand side as we let $N \to \infty$. In order to do that we will prove a few technical lemmas. The following lemma provides an alternative representation for the integrand in (6.8) such that only one orthogonal polynomial of degree N is present.

Lemma 6.7. *For each $t \in (0,1)$ we have*

$$\partial_t \log \prod_{j=0}^{N-1} \chi_j(t)^{-2} =$$

$$\int_{\mathbb{T}} \left[-N|p_N(z;t)|^2 + z\overline{p_N(z;t)}\partial_t p_N(z;t) + \overline{z} p_N(z;t)\overline{\partial_z p_N(z;t)} \right] V(z) e^{tV(z)} \frac{\mathrm{d}z}{2\pi \mathbb{i} z}.$$

Proof. Differentiating gives us

$$\partial_t \log \prod_{j=0}^{N-1} \chi_j(t)^{-2} = \partial_t \left(-2 \sum_{j=0}^{N-1} \log \chi_j(t) \right) = -2 \sum_{j=0}^{N-1} \frac{\partial_t \chi_j(t)}{\chi_j(t)}.$$

Then we use orthogonality to conclude that this equals

$$\begin{aligned} & -2 \sum_{j=0}^{N-1} \left(\partial_t \chi_j(t) \int_{\mathbb{T}} p_j(z;t) \overline{z}^j e^{tV(z)} \frac{\mathrm{d}z}{2\pi \mathbb{i} z} \right) \\ = & -2 \sum_{j=0}^{N-1} \int_{\mathbb{T}} p_j(z;t) \partial_t \overline{p_j(z;t)} e^{tV(z)} \frac{\mathrm{d}z}{2\pi \mathbb{i} z}. \end{aligned}$$

On the assumption $\chi_j(t) > 0$, this can be written as

$$
\begin{aligned}
&-\sum_{j=0}^{N-1} \int_{\mathbb{T}} \partial_t \left(p_j(z;t)\overline{p_j(z;t)} \right) e^{tV(z)} \frac{\mathrm{d}z}{2\pi \mathbb{i} z} \\
&= - \int_{\mathbb{T}} \left(\partial_t \sum_{j=0}^{N-1} |p_j(z;t)|^2 \right) e^{tV(z)} \frac{\mathrm{d}z}{2\pi \mathbb{i} z} \\
&= \int_{\mathbb{T}} \sum_{j=0}^{N-1} |p_j(z;t)|^2 V(z) e^{tV(z)} \frac{\mathrm{d}z}{2\pi \mathbb{i} z},
\end{aligned} \tag{6.9}
$$

where the last equality holds because according to basic differentiation rules, we have

$$
\begin{aligned}
&\int_{\mathbb{T}} \partial_t \left(\sum_{j=0}^{N-1} |p_j(z;t)|^2 e^{tV(z)} \right) \frac{\mathrm{d}z}{2\pi \mathbb{i} z} \\
&= \int_{\mathbb{T}} \left(\partial_t \sum_{j=0}^{N-1} |p_j(z;t)|^2 \right) e^{tV(z)} \frac{\mathrm{d}z}{2\pi \mathbb{i} z} + \int_{\mathbb{T}} \sum_{j=0}^{N-1} |p_j(z;t)|^2 V(z) e^{tV(z)} \frac{\mathrm{d}z}{2\pi \mathbb{i} z},
\end{aligned}
$$

where the left-hand side can be written as

$$
\sum_{j=0}^{N-1} \partial_t \int_{\mathbb{T}} |p_j(z;t)|^2 e^{tV(z)} \frac{\mathrm{d}z}{2\pi \mathbb{i} z} = 0
$$

with the help of the dominated convergence theorem and the fact $\langle p_j, p_j \rangle = 1$ for all j and t.

The Christoffel–Darboux identity, proven in [5, Lemma 2.3], states that for $|z| = 1$, we have

$$
\sum_{j=0}^{N-1} |p_j(z;t)|^2 = -N|p_N(z;t)|^2 + z\overline{p_N(z;t)}\partial_z p_N(z;t) + \overline{z} p_N(z;t)\overline{\partial_z p_N(z;t)}.
$$

Applying the identity to (6.9) completes the proof. □

The following lemma gives yet another representation for the expression $\partial_t \log \prod_{j=0}^{N-1} \chi_j(t)^{-2}$. This time we do not have any direct references to any orthogonal polynomials, but we express it with the help of the function Y defined in Lemma 6.6.

Lemma 6.8. *For each $t \in (0,1)$, we have*

$$
\partial_t \log \prod_{j=0}^{N-1} \chi_j(t)^{-2} = \oint_{\mathbb{T}} z^{-N} \left[Y(z;t)^{-1} \partial_z Y(z;t) \right]_{21} V(z) e^{tV(z)} \frac{\mathrm{d}z}{2\pi \mathbb{i}}
$$

Proof. We have the following recursion [5, Lemma 2.2, (2.4)]:

$$\chi_{N-1}(t)z^{N-1}\overline{p}_{N-1}(z^{-1};t) = z^N\chi_N(t)\overline{p}_N(z^{-1};t) - \overline{p}_N(0;t)p_N(z;t). \quad (6.10)$$

We showed earlier that we have $\det Y(z;t) = 1$. With the help of this we compute

$$\begin{aligned}[Y(z;t)^{-1}\partial_z Y(z;t)]_{21} =& - Y_{21}(z;t)\partial_z Y_{11}(z;t) + Y_{11}(z;t)\partial_z Y_{21}(z;t) \\ =& \frac{\chi_{N-1}(t)}{\chi_N(t)} z^{N-1} p^*_{N-1}(z^{-1};t)\partial_z p_N(z;t) \\ & - \frac{\chi_{N-1}(t)}{\chi_N(t)} p_N(z;t)\partial_z\left(z^{N-1}p^*_{N-1}(z^{-1})\right).\end{aligned}$$

Using (6.10), it is possible to write this as

$$\begin{aligned}[Y(z;t)^{-1}\partial_z Y(z;t)]_{21} =& \left[z^N p^*_N(z^{-1};t) - \frac{p^*_N(0;t)}{\chi_N(t)}p_N(z;t)\right]\partial_z p_N(z;t) \\ & - p^*_N(z;t)\partial_z\left[z^N p^*_N(z^{-1};t) - \frac{p^*_N(0;t)}{\chi_N(t)}p_N(z;t)\right] \\ =& z^N p^*_N(z^{-1};t)\partial_z p_N(z;t) - Nz^{N-1}p_N(z;t)p^*_N(z^{-1};t) \\ & - z^N p_N(z;t)\partial_z;t)\partial_z p^*_N(z^{-1};t).\end{aligned}$$

If we assume $|z| = 1$, then we have $p^*_N(z^{-1};t) = \overline{p_N(z;t)}$, and we write

$$\begin{aligned}&[Y(z;t)^{-1}\partial_z Y(z;t)]_{21} = \\ &\qquad z^{N-1}\left(\overline{p_N(z;t)}\partial_z p_N(z;t) + z^{-1}p_N(z;t)\overline{\partial_z p_N(z;t)} - N|p_N(z;t)|^2\right).\end{aligned}$$

Multiplication by z^{-N} and plugging into Lemma 6.7 completes the proof. □

The following lemma gives us a representation for $\partial_t \log\prod_{j=0}^{N-1}\chi_j(t)^{-2}$ using the inverse and derivative of the function Y. Note that we assume that the function V has an analytic continuation into some annulus surrounding the unit circle.

Lemma 6.9. *Let $\varepsilon > 0$ such that the circles $|z| = 1\pm\varepsilon$ are within the annulus $\{z \in \mathbb{C} : ||z|-1| < r\}$, into which V has an analytic continuation. Then, for each $t \in (0,1)$, we have*

$$\begin{aligned}\partial_t \log\prod_{j=0}^{N-1}\chi_j(t)^{-2} =& \oint_{|z|=1+\varepsilon}\left[Y(z;t)^{-1}\partial_z Y(z;t)\right]_{11}V(z)\frac{\mathrm{d}z}{2\pi\mathbb{i}} \\ & - \oint_{|z|=1-\varepsilon}\left[Y(z;t)^{-1}\partial_z Y(z;t)\right]_{11}V(z)\frac{\mathrm{d}z}{2\pi\mathbb{i}}\end{aligned}$$

Proof. It was shown in Lemma 6.5 that $Y(z;t)$ is a solution to a Riemann-Hilbert problem, having continuous boundary values Y_+ and Y_- from the inside and from the outside of the unit circle, respectively. These limits are related by the jump condition

$$Y_+(z;t) = Y_-(z;t)\begin{pmatrix} 1 & z^{-N}e^{tV(z)} \\ 0 & 1 \end{pmatrix}.$$

Using this, we obtain through direct computation:

$$\begin{aligned}
&\left[Y(z;t)^{-1}\partial_z Y(z;t)\right]_{11,+} - \left[Y(z;t)^{-1}\partial_z Y(z;t)\right]_{11,-} \\
=&Y_{22,+}(z;t)\partial_z Y_{11}(z;t) - Y_{12,+}(z;t)\partial_z Y_{21}(z;t) \\
&- Y_{22,-}(z;t)\partial_z Y_{11}(z;t) + Y_{12,-}(z;t)\partial_z Y_{21}(z;t) \\
=&\left(Y_{22,-}(z;t) + z^{-N}e^{tV(z)}Y_{21}(z;t)\right)\partial_z Y_{11}(z;t) \\
&- \left(Y_{12,-}(z;t) + z^{-N}e^{tV(z)}Y_{11}(z;t)\right)\partial_z Y_{21}(z;t) \\
&\quad - Y_{22,-}(z;t)\partial_z Y(z;t) + Y_{12,-}(z;t)\partial_z Y(z;t) \\
=&z^{-N}e^{tV(z)}[Y_{21}(z;t)\partial_z Y_{11}(z;t) - Y_{11}(z;t)\partial_z Y_{21}(z;t)] \\
=&-z^{-N}e^{tV(z)}[Y(z;t)^{-1}\partial_z Y(z;t)]_{21}.
\end{aligned}$$

Plugging this into Lemma 6.8 yields

$$\begin{aligned}
&\partial_t \log \prod_{j=0}^{N-1} \chi_j(t)^{-2} \\
=&\oint_{\mathbb{T}} \left([Y(z;t)^{-1}\partial_z Y(z;t)]_{11,-} - [Y(z;t)^{-1}\partial_z Y(z;t)]_{11,+}\right) V(z)\frac{\mathrm{d}z}{2\pi \mathbb{i}}.
\end{aligned}$$

The function $Y(z;t)$ is holomorphic in $\mathbb{C} \setminus \mathbb{T}$ and $Y_+(z)$ is the limit when approaching the unit circle from the inside. Also, the function V is holomorphic in an annulus around $\mathbb{T}$, so we can use the Cauchy integral theorem to conclude that

$$\oint_{\mathbb{T}} [Y(z;t)^{-1}\partial_z Y(z;t)]_{11,\pm} V(z)\frac{\mathrm{d}z}{2\pi \mathbb{i}} = \oint_{|z|=1\mp\varepsilon} [Y(z;t)^{-1}\partial_z Y(z;t)]_{11} V(z)\frac{\mathrm{d}z}{2\pi \mathbb{i}}$$

holds. This competes the proof.

□

Next, we want to determine the asymptotic properties of the right-hand side of the identity in Lemma 6.9.

Let us study some properties of the function $Y(\,\cdot\,;t)$. For all $t \in [0,1]$, $Y(\,\cdot\,;t)$ is a holomorphic function $\mathbb{C}\setminus\mathbb{T} \to \mathbb{C}^{2\times 2}$. It has continuous boundary values $Y_\pm$ on the unit circle satisfying

$$Y_+(z;t) = Y_-(z;t)\begin{pmatrix} 1 & z^N e^{tV(z)} \\ 0 & 1 \end{pmatrix}.$$

Also, as $|z| \to \infty$, we have

$$Y(z;t) = \left(I + O\left(z^{-1}\right)\right)\begin{pmatrix} z^N & 0 \\ 0 & z^{-N} \end{pmatrix}.$$

These properties simply follow from Lemma 6.6 by choosing $tV(z)$ as the weight function.

In order to derive formulas for the asymptotics of the integrals that appear in Lemma 6.9, we will define some auxiliary functions that are constructed by modifying the function $Y(\,\cdot\,;t)$. This will allow us to divide the jump on the unit circle into multiple parts, which makes the computations easier.

Definition 6.3. *Let us define a function* $T : \mathbb{C}\setminus\mathbb{T} \to \mathbb{C}^{2\times 2}$ *as*

$$T(z;t) = \begin{cases} Y(z;t), & |z| < 1 \\ Y(z;t)\begin{pmatrix} z^{-N} & 0 \\ 0 & z^N \end{pmatrix}, & |z| > 1. \end{cases}$$

The following lemma shows some important properties of the function $T(\cdot,t)$. It is holomorphic in its domain as was $Y(\cdot;t)$, but the jump condition for the limits on the unit circle, and asymptotics in the infinity are different.

Lemma 6.10. *The function* $T(\,\cdot\,;t)$ *has the following properties:*

1. $T(\,\cdot\,;t) : \mathbb{C}\setminus\mathbb{T} \to \mathbb{C}^{2\times 2}$ *is holomorphic*
2. $T(\,\cdot\,;t)$ *has continuous boundary values on* $\mathbb{T}$ *satifying*

$$T_+(z;t) = T_-(z;t)\begin{pmatrix} z^N & e^{tV(z)} \\ 0 & z^{-N} \end{pmatrix}$$

3. $T(z,t) = I + O\left(z^{-1}\right)$, *as* $z \to \infty$.

Proof. The first claim follows directly from the properties of Y.

For the second claim, we perform a direct computation

$$\begin{aligned} T_+(z;t) = Y_+(z;t) &= Y_-(z;t)\begin{pmatrix}1 & z^{-N}e^{tV(z)}\\ 0 & 1\end{pmatrix} \\ &= Y_-(z;t)\begin{pmatrix}z^{-N} & 0\\ 0 & z^{N}\end{pmatrix}\begin{pmatrix}z^{N} & 0\\ 0 & z^{-N}\end{pmatrix}\begin{pmatrix}1 & z^{-N}e^{tV(z)}\\ 0 & 1\end{pmatrix} \\ &= T_-(z;t)\begin{pmatrix}z^{N} & e^{tV(z)}\\ 0 & z^{-N}\end{pmatrix}. \end{aligned}$$

For part 3, we can see that for $|z| > 1$, we have

$$\begin{aligned} T(z;t) &= Y(z;t)\begin{pmatrix}z^{-N} & 0\\ 0 & z^{N}\end{pmatrix} \\ &= \left(I + O\left(z^{-1}\right)\right)\begin{pmatrix}z^{N} & 0\\ 0 & z^{-N}\end{pmatrix}\begin{pmatrix}z^{-N} & 0\\ 0 & z^{N}\end{pmatrix} \\ &= I + O\left(z^{-1}\right). \end{aligned}$$

□

Next, we use the function T to define another auxiliary function.

Definition 6.4. *Let $R > 0$ and V holomorphic in the annulus $\{||z|-1| < R\}$. Let $0 < r < \varepsilon < R$. Let us define*

$$S(z;t) = \begin{cases} T(z;t), & |z| > 1 + r \text{ and } |z| < 1 - r \\ T(z;t)\begin{pmatrix}1 & 0\\ z^{-N}e^{-tV(z)} & 1\end{pmatrix}, & 1 < |z| < 1 + r \\ T(z;t)\begin{pmatrix}1 & 0\\ -z^{N}e^{-tV(z)} & 1\end{pmatrix}, & 1 - r < |z| < 1 \end{cases}$$

The following lemma shows that this function has continuous boundary values on three concentric circles, satisfying certain jump conditions.

Lemma 6.11. *For $\widetilde{\Gamma} = \mathbb{T} \cup ((1+r)\mathbb{T}) \cup ((1-r)\mathbb{T})$, for all $t \in [0,1]$, $S(\,\cdot\,;t)$ has the following properties:*

1. *$S(\,\cdot\,;t)$ has continuous boundary values $S_\pm$ on $\widetilde{\Gamma}$ such that they satisfy*

$$S_+(z;t) = \begin{cases} S_-(z;t)\begin{pmatrix}1 & 0\\ z^{-N}e^{-tV(z)} & 1\end{pmatrix}, & |z| = 1 + r \\ S_-(z;t)\begin{pmatrix}0 & e^{tV(z)}\\ -e^{-tV(z)} & 0\end{pmatrix}, & |z| = 1 \\ S_-(z;t)\begin{pmatrix}1 & 0\\ z^{N}e^{-tV(z)} & 1\end{pmatrix}, & |z| = 1 - r \end{cases}$$

2. We have

$$S(z;t) = I + O\left(z^{-1}\right), \text{ as } |z| \to \infty.$$

Proof. Let $|z| = 1 + r$. Then $S_-(z;t) = T(z;t)$ and

$$S_+(z;t) = T(z;t) \begin{pmatrix} 1 & 0 \\ z^{-N} e^{-tV(z)} & 1 \end{pmatrix} = S_-(z;t) \begin{pmatrix} 1 & 0 \\ z^{-N} e^{-tV(z)} & 1 \end{pmatrix}.$$

For $|z| = 1$, we use the jump condition from Lemma 6.10. We can write

$$\begin{aligned} S_+(z;t) &= T_+(z;t) \begin{pmatrix} 1 & 0 \\ -z^{-N} e^{-tV(z)} & 1 \end{pmatrix} \\ &= T_-(z;t) \begin{pmatrix} z^N & e^{tV(z)} \\ 0 & z^{-N} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z^{-N} e^{-tV(z)} & 1 \end{pmatrix}. \end{aligned}$$

We multiply this by an expression that equals the identity matrix I, and then we compare the result with the definition of $S(z;t)$.

$$\begin{aligned} &T_-(z;t) \begin{pmatrix} 1 & 0 \\ z^{-N} e^{-tV(z)} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -z^{-N} e^{-tV(z)} & 1 \end{pmatrix} \begin{pmatrix} z^N & e^{tV(z)} \\ 0 & z^{-N} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z^{-N} e^{-tV(z)} & 1 \end{pmatrix} \\ =&S_-(z;t) \begin{pmatrix} 0 & e^{tV(z)} \\ e^{-tV(z)} & 0 \end{pmatrix}. \end{aligned}$$

For $|z| = 1 - r$, we have $S_+(z;t) = T(z;t)$. We multiply this by something that equals the identity matrix, and obtain

$$\begin{aligned} S_+(z;t) &= T(z;t) \begin{pmatrix} 1 & 0 \\ -z^{N} e^{-tV(z)} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z^{N} e^{-tV(z)} & 1 \end{pmatrix} \\ &= S_-(z;t) \begin{pmatrix} 1 & 0 \\ z^{N} e^{-tV(z)} & 1 \end{pmatrix}. \end{aligned}$$

Part 2 follows directly from Lemma 6.10 and the fact $S(z;t) = T(z;t)$ for large values of $|z|$. $\square$

We define yet another auxiliary function.

Definition 6.5. *Let us define the fuction $P : \mathbb{C} \setminus \mathbb{T} \to \mathbb{C}^{2\times 2}$ as*

$$P(z;t) = \begin{cases} \begin{pmatrix} e^{t \oint_{\mathbb{T}} \frac{V(\omega)}{\omega - z} \frac{d\omega}{2\pi i}} & 0 \\ 0 & e^{-t \oint_{\mathbb{T}} \frac{V(\omega)}{\omega - z} \frac{d\omega}{2\pi i}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & |z| < 1 \\ \begin{pmatrix} e^{t \oint_{\mathbb{T}} \frac{V(\omega)}{\omega - z} \frac{d\omega}{2\pi i}} & 0 \\ 0 & e^{-t \oint_{\mathbb{T}} \frac{V(\omega)}{\omega - z} \frac{d\omega}{2\pi i}} \end{pmatrix}, & |z| > 1. \end{cases}$$

This function is again holomorphic in its domain, and it satisfies a jump condition on the unit circle, as the following lemma shows.

Lemma 6.12. *For each $t \in [0,1]$, the function $P(\cdot\,;t)$ satisfies*

1. *$P(\,\cdot\,;t) : \mathbb{C} \setminus \mathbb{T} \to \mathbb{C}^{2\times 2}$ is holomorphic,*
2. *$P(\,\cdot\,;t)$ has continuous boundary values $P_\pm(\,\cdot\,;t)$ such that*

$$P_+(z;t) = P_-(z;t) \begin{pmatrix} 0 & e^{tV(z)} \\ -e^{-tV(z)} & 0 \end{pmatrix}, |z| = 1$$

3. *$P(z;t) = I + O\left(z^{-1}\right)$ as we let $|z| \to \infty$.*

Proof. For part 1, P is holomorphic because all of the matrix elements are compositions of holomorphic functions.

In order to prove part 2, we assume $|z| = 1$, and compute

$$\begin{aligned} P_-(z;t)^{-1}P_+(z;t) &= \begin{pmatrix} e^{-tJ_{V,-}(z)} & 0 \\ 0 & e^{tJ_{V,-}(z)} \end{pmatrix} \begin{pmatrix} e^{tJ_{V,+}(z)} & 0 \\ 0 & e^{-tJ_{V,+}(z)} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} e^{t\left(J_{V,+}(z)-J_{V,-}(z)\right)} & 0 \\ 0 & e^{-t\left(J_{V,+}(z)-J_{V,-}(z)\right)} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned}$$

According to the equation (6.3), we have $J_{V,+}(z) - J_{V,-}(z) = V(z)$, so the right-hand side simplifies to

$$\begin{pmatrix} e^{tV(z)} & 0 \\ 0 & e^{-tV(z)} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & e^{tV(z)} \\ -e^{-tV(z)} & 0 \end{pmatrix}.$$

Multiplying from the left by $P_-(z;t)$ completes the proof.

For part 3, we note $\oint_{\mathbb{T}} \frac{V(\omega)}{\omega - z}\frac{\mathrm{d}\omega}{2\pi \mathrm{i}} = O(z^{-1})$, and write

$$P(z;t) = \begin{pmatrix} e^{O(z^{-1})} & 0 \\ 0 & e^{O(z^{-1})} \end{pmatrix} = I + O\left(z^{-1}\right).$$

□

Let us define one more auxiliary function with the help of the previous functions that we have defined.

Definition 6.6. *Let us define*

$$R(z;t) = S(z;t)P(z;t)^{-1}$$

The following lemma shows the jump conditions and asymptotic behavior of the function.

Lemma 6.13. *Let $\Gamma = ((1+r)\mathbb{T}) \cup ((1-r)\mathbb{T})$. For each $t \in [0,1]$, $R(\,\cdot\,;t)$ has a holomorphic continuation into $\mathbb{C} \setminus \Gamma$ such that $R(\,\cdot\,;t)$ is a solution to the problem*

1. *$R(\,\cdot\,;t) : \mathbb{C} \setminus \Gamma \to \mathbb{C}^{2\times 2}$ is holomorphic*
2. *R has continuous boundary values on Γ and there is a value c that is independent of N and t such that*

   $$R_+(z;t) = R_-(z;t)\,(I + \Delta_N)\,,$$

   where $|z| = 1 \pm r$ and $\Delta_N = O\left(e^{-cN}\right)$.
3. *$R(z;t) = I + O\left(z^{-1}\right),\ |z| \to \infty.$*

Proof. It is trivial that $R(z;t)$ is holomorphic for $|z| \neq 1$ and $z \notin \Gamma$. What remains is to show that it is holomorphic on the unit circle. This is equivalent to showing that we have $R_+(z;t) = R_-(z;t)$, like in the argument for $\det(Y)$ in Lemma 6.5. Using the definition of R and the previous lemmas we can see that

$$\begin{aligned} R_+(z;t) &= S_+(z;t)P_+(z;t)^{-1} \\ &= S_-(z;t)\begin{pmatrix} 0 & e^{tV(z)} \\ -e^{-tV(z)} & 0 \end{pmatrix}\left(P_-(z;t)\begin{pmatrix} 0 & e^{tV(z)} \\ -e^{-tV(z)} & 0 \end{pmatrix}\right)^{-1} \\ &= S_-(z;t)P_-(z;t)^{-1} = R_-(z;t). \end{aligned}$$

For part 2, let us first assume $|z| = 1 + r$. Now, using Lemma 6.11, we obtain

$$R_+(z;t) = S_+(z;t)P(z;t)^{-1} = S_-(z;t)\begin{pmatrix} 1 & 0 \\ z^{-N}e^{-tV(z)} & 1 \end{pmatrix}P(z;t)^{-1}.$$

We have $S_-(z;t) = R_-(z;t)P(z;t)$ from the definition of $R(z;t)$. Plugging this in yields

$$R_+(z;t) = R_-(z;t)P(z;t)\begin{pmatrix} 1 & 0 \\ z^{-N}e^{-tV(z)} & 1 \end{pmatrix}P(z;t)^{-1} = I + O\left(e^{-cN}\right).$$

The proof for the case $|z| = 1 - r$ follows the same steps.

Part 3 follows directly from Lemma 6.11 and Lemma 6.12. □

Let us define norms for matrices and matrix-valued functions that we are dealing with. For the matrices, we will use the Hilbert–Schmidt norm defined as

$$\|A\|_{\mathrm{mat}} = \sqrt{\sum_{i,j} |a_{ij}|^2},$$

where a_{ij} are the matrix elements of A. This norm is submultiplicative, i.e. it satisfies $\|AB\|_{\mathrm{mat}} \leq \|A\|_{\mathrm{mat}} \|B\|_{\mathrm{mat}}$ for all $A, B \in \mathbb{C}^{k\times k}$, which can be shown with the help of the Cauchy–Schwarz inequality.

For the $\mathbb{C}^{2\times 2}$-valued functions, we use the L^2-norm defined as

$$\|f\|_{L^2_{\mathrm{mat}}(\Gamma)} = \sqrt{\int_\Gamma \|f(z)\|^2_{\mathrm{mat}} |\mathrm{d}z|}.$$

The L^∞-norm is defined as

$$L^\infty_{\mathrm{mat}} = \sup_{z\in\Gamma} \|f(z)\|^2_{\mathrm{mat}}.$$

For operators $L^2(\Gamma) \to L^2(\Gamma)$, we use the standard operator norm defined as

$$\|A\|_{\mathrm{op}} = \sup_{\|f\|\leq 1} \|A(f)\|_{L^2_{\mathrm{mat}}(\Gamma)}.$$

The following proposition gives an upper bound for the operator norm of an operator that is of a certain form. This technical result will be useful when further analyzing the properties of the function R near Γ.

Proposition 6.1. *Let A be an operator $L^2(\Gamma) \to L^2(\Gamma)$ such that we have $\|A\|_{op} < 1 - \delta$ for some $\delta > 0$. The operator $(\mathrm{Id} + A)^{-1}$, $L^2(\Gamma) \to L^2(\Gamma)$ exists and satisfies*

$$\left\|(\mathrm{Id} + A)^{-1} - \mathrm{Id}\right\|_{op} \leq \frac{1}{\delta} \|A\|_{op}.$$

Proof. Since we assume $\|A\| < 1 - \delta$, it is possible to express $(\mathrm{Id} + A)^{-1}$ as a Neumann series [20, Chapter IV, Theorem 1.4]

$$(\mathrm{Id} + A)^{-1} = \sum_{k=0}^{\infty} (-A)^k.$$

Using this series expansion, we obtain

$$\begin{aligned}\left\|(\mathrm{Id}+A)^{-1}-\mathrm{Id}\right\|_{\mathrm{op}} &= \left\|\sum_{k=0}^{\infty}(-A)^k-\mathrm{Id}\right\|_{\mathrm{op}} = \left\|-A\sum_{k=0}^{\infty}(-A)^k\right\|_{\mathrm{op}} \\ &\leq \|A\|_{\mathrm{op}}\sum_{k=0}^{\infty}\|A\|_{\mathrm{op}}^k = \frac{\|A\|_{\mathrm{op}}}{1-\|A\|_{\mathrm{op}}} \\ &\leq \|A\|_{\mathrm{op}}\cdot\frac{1}{1-(1-\delta)} = \frac{1}{\delta}\|A\|_{\mathrm{op}}.\end{aligned}$$

□

The following lemma will play a part in showing boundedness of a certain operator later. It is similar to the result in Lemma 6.3, but we need to show that it also applies to the matrix-valued functions that we are using.

Lemma 6.14. *Let f be a $\mathbb{C}^{2\times 2}$-valued function in $L^2(\mathbb{T})$. Then, we have*

$$\int_{\mathbb{T}}\left\|\oint_{\mathbb{T}}\frac{f(\omega)}{\omega-z}\frac{\mathrm{d}\omega}{2\pi\mathbb{i}}\right\|_{mat}^2|\mathrm{d}z| \leq \int_{\mathbb{T}}\|f(z)\|_{mat}^2\,|\mathrm{d}z|.$$

Proof. Let us write

$$\begin{aligned} f &= \begin{pmatrix} f_{11} & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix} 0 & f_{12}\\ 0 & 0\end{pmatrix} + \begin{pmatrix} 0 & 0\\ f_{21} & 0\end{pmatrix} + \begin{pmatrix} 0 & 0\\ 0 & f_{22}\end{pmatrix} \\ &= f_1+f_2+f_3+f_4.\end{aligned}$$

This notation allows us to write

$$\int_{\mathbb{T}}\left\|\oint_{\mathbb{T}}\frac{f(\omega)}{\omega-z}\frac{\mathrm{d}\omega}{2\pi\mathbb{i}}\right\|_{\mathrm{mat}}^2|\mathrm{d}z| \leq \sum_{j=1}^{4}\int_{\mathbb{T}}\left\|\oint_{\mathbb{T}}\frac{f_j(\omega)}{\omega-z}\frac{\mathrm{d}\omega}{2\pi\mathbb{i}}\right\|_{\mathrm{mat}}^2|\mathrm{d}z|.$$

Since we have $\|f_1\|_{\mathrm{mat}}^2 = |f_{11}|^2$ etc., we conclude that the above is equal to

$$\begin{aligned}\sum_{i,j}\int_{\mathbb{T}}\left|\oint_{\mathbb{T}}\frac{f_{ij}(\omega)}{\omega-z}\frac{\mathrm{d}\omega}{2\pi\mathbb{i}}\right|^2|\mathrm{d}z| &\leq \sum_{i,j}\int_{\mathbb{T}}|f_{ij}(z)|^2\,|\mathrm{d}z| \\ &= \int_{\mathbb{T}}\|f(z)\|_{\mathrm{mat}}^2\,|\mathrm{d}z| = \|f\|_{L^2_{\mathrm{mat}}(\mathbb{T})}^2,\end{aligned}$$

where the inequality comes from Lemma 6.3.

□

Next, we define an operator that will be useful when analyzing the properties of the function R. We prove that the operator is bounded.

Lemma 6.15. *Let $f : \Gamma \to \mathbb{C}^{2\times 2}$ be a function in $L^2(\Gamma)$. Let us define an operator $I^{\Gamma}_{\cdot,-} : L^2_{mat}(\Gamma) \to L^2_{mat}(\Gamma)$ as*

$$I^{\Gamma}_{f,-}(z) = \lim_{z\to\Gamma} \oint_{\Gamma} \frac{f(\omega)}{\omega - z}\frac{\mathrm{d}\omega}{2\pi\mathbb{i}},$$

where the limit is considered as approaching Γ from the inside in a non-tangential manner. The operator $I^{\Gamma}_{f,-}$ is bounded.

Proof. Let $\Gamma_{\pm} = \{|z| = 1 \pm r\}$. Then we can write $\Gamma = \Gamma_+ \cup \Gamma_-$ and

$$\begin{aligned}
\|I^{\Gamma}_{f}(z)\|_{L^2_{\mathrm{mat}}(\Gamma)} =& \int_{\Gamma} \left\| \oint_{\Gamma_+} \frac{f(\omega)}{\omega - z}\frac{\mathrm{d}\omega}{2\pi\mathbb{i}} + \oint_{\Gamma_-} \frac{f(\omega)}{\omega - z}\frac{\mathrm{d}\omega}{2\pi\mathbb{i}} \right\|^2_{\mathrm{mat}} |\mathrm{d}z| \\
\leq& 4\left(\int_{\Gamma_+} \left\| \oint_{\Gamma_+} \frac{f(\omega)}{\omega - z}\frac{\mathrm{d}\omega}{2\pi\mathbb{i}} \right\|^2_{\mathrm{mat}} |\mathrm{d}z| + \int_{\Gamma_-} \left\| \oint_{\Gamma_-} \frac{f(\omega)}{\omega - z}\frac{\mathrm{d}\omega}{2\pi\mathbb{i}} \right\|^2_{\mathrm{mat}} |\mathrm{d}z| \right. \\
&\left. + \int_{\Gamma_+} \left\| \oint_{\Gamma_-} \frac{f(\omega)}{\omega - z}\frac{\mathrm{d}\omega}{2\pi\mathbb{i}} \right\|^2_{\mathrm{mat}} |\mathrm{d}z| + \int_{\Gamma_-} \left\| \oint_{\Gamma_+} \frac{f(\omega)}{\omega - z}\frac{\mathrm{d}\omega}{2\pi\mathbb{i}} \right\|^2_{\mathrm{mat}} |\mathrm{d}z| \right).
\end{aligned}$$

There are four terms. The last two terms can be approximated by

$$\int_{\Gamma_\pm} \left\| \oint_{\Gamma_\mp} \frac{f(\omega)}{\omega - z}\frac{\mathrm{d}\omega}{2\pi\mathbb{i}} \right\|^2_{\mathrm{mat}} |\mathrm{d}z| \leq C_1 \int_{\Gamma_\pm} \left\| \oint_{\Gamma_\mp} f(\omega)\frac{\mathrm{d}\omega}{2\pi\mathbb{i}} \right\|^2_{\mathrm{mat}} |\mathrm{d}z| \leq C_2 \|f\|_{L^2_{\mathrm{mat}}},$$

where the constant

$$C_1 = \sup_{\omega\in\Gamma_+, z\in\Gamma_-} |\omega - z|^{-1}$$

is finite because the integrations are performed along concentric circles of different diameters.

The terms where the integrals are along the same circle can be scaled using a change of variables. Let us define a constant ρ such that $\Gamma_+ = \rho\mathbb{T}$. Let us define $\widetilde{\omega} = \rho\omega$ and $\widetilde{z} = \rho z$. Also, we define a function $\widetilde{f}$ such that $\widetilde{f}(\xi) = f(\rho\xi)$ for all ξ. The change of variables yields

$$\int_{\Gamma_+} \left\| \oint_{\Gamma_+} \frac{f(\omega)}{\omega - z}\frac{\mathrm{d}\omega}{2\pi\mathbb{i}} \right\|^2_{\mathrm{mat}} |\mathrm{d}z| = \rho \int_{\mathbb{T}} \left\| \oint_{\mathbb{T}} \frac{\widetilde{f}(\widetilde{\omega})}{\widetilde{\omega} - \widetilde{z}}\frac{\mathrm{d}\widetilde{\omega}}{2\pi\mathbb{i}} \right\|^2_{\mathrm{mat}} |\mathrm{d}\widetilde{z}|.$$

By Lemma 6.14, we obtain

$$\rho \int_{\mathbb{T}} \left\| \oint_{\mathbb{T}} \frac{\widetilde{f}(\widetilde{\omega})}{\widetilde{\omega} - \widetilde{z}}\frac{\mathrm{d}\widetilde{\omega}}{2\pi\mathbb{i}} \right\|^2_{\mathrm{mat}} |\mathrm{d}\widetilde{z}| \leq \rho \|\widetilde{f}\|^2_{L^2_{\mathrm{mat}}(\mathbb{T})} = \rho \int_{\mathbb{T}} \|\widetilde{f}(\widetilde{\omega})\|^2_{\mathrm{mat}} |\mathrm{d}\widetilde{\omega}|.$$

A change of variables yields

$$\rho \int_{\mathbb{T}} \|\widetilde{f}(\widetilde{\omega})\|_{\mathrm{mat}}^2 |\mathrm{d}\widetilde{\omega}| = \int_{\rho\mathbb{T}} \|f(\omega)\|_{\mathrm{mat}}^2 |\mathrm{d}\omega| \leq \|f\|_{L^2_{\mathrm{mat}}(\Gamma)}^2.$$

The reasoning for the term

$$\int_{\Gamma_-} \left\| \oint_{\Gamma_-} \frac{f(\omega)}{\omega - z} \frac{\mathrm{d}\omega}{2\pi\mathbb{i}} \right\|_{\mathrm{mat}}^2 |\mathrm{d}z|$$

is identical. Hence, we have

$$\|I_{f,-}^{\Gamma}(z)\|_{L^2_{\mathrm{mat}}(\Gamma)} \leq C \|f\|_{L^2_{\mathrm{mat}}(\Gamma)}^2$$

for a constant C. Plugging any function f with $\|f\|_{L^2_{\mathrm{mat}}(\Gamma)}^2 = 1$ into the above inequality shows that the operator norm satisfies

$$\|I_-^{\Gamma}\|_{L^2_{\mathrm{mat}}(\Gamma) \to L^2_{\mathrm{mat}}(\Gamma)} \leq C.$$

□

Now we are ready to continue the analysis of the function R. We are especially interested in the asymptotics of the function for large values of N, which the following lemma analyzes.

Lemma 6.16. *For $|z| = 1 \pm \varepsilon$, $0 < \varepsilon < r$, we have*

$$R(z;t) = I + O\left(e^{-cN}\right)$$

and

$$\partial_z R(z;t) = O\left(e^{-cN}\right)$$

for some $c > 0$ independent of N, z and t.

Proof. Let us recall the jump condition from Lemma 6.16. In this proof, we use a shorthand $R(z) = R(z;t)$. According to Lemma 6.16, we have

$$R_+(z) = R_-(z)\left(I + O\left(e^{-cN}\right)\right) = R_-(z)\left(I + \Delta_N(z)\right).$$

Next, we define a function

$$\widehat{R}(z) = I + \oint_{\Gamma} \frac{R_-(\omega)\Delta_N(\omega)}{\omega - z} \frac{\mathrm{d}\omega}{2\pi\mathbb{i}},$$

where $z \notin \Gamma$, and the integration is done counter-clockwise along two circles of radii $1 \pm r$. Let us define an operator I_f^Γ as

$$I_f^\Gamma = \oint_\Gamma \frac{f(\omega)}{\omega - z} \frac{\mathrm{d}\omega}{2\pi \mathrm{i}}.$$

This enables us to write $\widehat{R} = I + I^\Gamma_{R_-\Delta_N}(z)$.

Note that I_f^Γ only differs from the Cauchy tranform I_f by the integration path. These operators share many properties, including the formula (6.3) but this time when approaching Γ. This can be shown by a similar argument to that we used with (6.3), but this time the integral is expressed as a sum of two integrals, one along $(1-r)\mathbb{T}$ and other along $(1+r)\mathbb{T}$. Scaling the variables such that the integrations are done on the unit circle, as we did in the proof of Lemma 6.15, allows the use of (6.3) as the extra terms that appeared due to scaling cancel each other. Thus, we have $I^\Gamma_{f,+} - I^\Gamma_{f,-} = f$.

Let us consider the limits when approaching Γ from different sides. They can be written as

$$\begin{aligned}
\widehat{R}_+(z) - \widehat{R}_-(z) &= I^\Gamma_{R_-\Delta_N,+}(z) - I^\Gamma_{R_-\Delta_N,-}(z) \\
&= R_-(z)\Delta_N(z) && \text{(formula (6.3))} \\
&= R_+(z) - R_-(z). && \text{(the jump condition for } R\text{)}
\end{aligned}$$

Thus, the function $\widehat{R} - R$ has no jump on Γ, so it is holomorphic in $\mathbb{C}$ by a similar argument to the one in the proof for $\det Y$ in the 2nd part of Lemma 6.5. Moreover, $\widehat{R}(z) - R(z) = O\left(\frac{1}{z}\right)$ as $|z| \to \infty$. Using Liouville's theorem we conclude that $\widehat{R} - R = 0$, which implies

$$R(z) = I + \oint_\Gamma \frac{R_-(\omega)\Delta_N(\omega)}{\omega - z} \frac{\mathrm{d}\omega}{2\pi \mathrm{i}}.$$

Letting $z \to \Gamma$ from the $-$-side yields

$$R_-(z) = I + I_-^\Gamma(R_-\Delta_N). \tag{6.11}$$

Let us define an operator $L_N : L^2_{\mathrm{mat}}(\Gamma) \to L^2_{\mathrm{mat}}(\Gamma)$ as

$$L_N(f) = I_-^\Gamma(f\Delta_N). \tag{6.12}$$

This definition allows us to write (6.11) as

$$(\mathrm{Id} - L_N)R_- = I. \tag{6.13}$$

It is useful to study the properties of operator L_N, especially its norm.

Using (6.12), we can see that for a function f, we have

$$\|L_N(f)\|_{L^2_{\text{mat}}} = \|I^{\Gamma}_{-}(f\Delta_N)\|_{L^2_{\text{mat}}}.$$

Since we have $\|I^{\Gamma}_{-}\| \leq C'$ where C' is some constant independent of N, we have

$$\|I^{(\Gamma)}_{-}(f\Delta_N)\|_{L^2_{\text{mat}}} \leq C'\|f\Delta_N\|_{L^2_{\text{mat}}} = C'\left(\int_{\Gamma}\|f\Delta_N\|_{\text{mat}}|\mathrm{d}z|\right)^{\frac{1}{2}}.$$

The sub-multiplicative property of the matrix norm implies

$$\begin{aligned} C'\left(\int_{\Gamma}\|f\Delta_N\|^2_{\text{mat}}|\mathrm{d}z|\right)^{\frac{1}{2}} &\leq C'\left(\int_{\Gamma}\|f\|^2_{\text{mat}}\|\Delta_N\|^2_{\text{mat}}|\mathrm{d}z|\right)^{\frac{1}{2}} \\ &\leq C'\|\Delta_N\|_{L^{\infty}_{\text{mat}}}\|f\|_{L^2_{\text{mat}}}. \end{aligned}$$

Since we have $\|\Delta_N\|_{L^{\infty}_{\text{mat}}} = O(e^{-cN})$, we can choose N large enough such that $\|\Delta_N\|_{L^{\infty}_{\text{mat}}} < 1$ is satisfied. Thus, we have $\|L_N\|_{L^2_{\text{mat}}\to L^2_{\text{mat}}} < 1$.

This implies that with a sufficiently large N, the operator $(\mathrm{Id} - L_N)$ has an inverse that can be expressed as a Neumann series $(\mathrm{Id} - L_N)^{-1} = \sum_{k=0}^{\infty} L_N^k$. The existence of the inverse implies that (6.13) is equivalent to

$$R_- = (\mathrm{Id} - L_N)^{-1}(I).$$

Subtracting the identity matrix I from both sides yields

$$R_- - I = \left[(\mathrm{Id} - L_N)^{-1} - \mathrm{Id}\right](I).$$

We take the norms on both sides and obtain

$$\|R_- - I\|_{L^2_{\text{mat}}} = \left\|\left[(\mathrm{Id} - L_N)^{-1} - \mathrm{Id}\right](I)\right\|_{L^2_{\text{mat}}} \leq \left\|(\mathrm{Id} - L_N)^{-1} - \mathrm{Id}\right\|_{L^2_{\text{mat}}\to L^2_{\text{mat}}}.$$

Applying Proposition 6.1 shows that we have

$$\|R_- - I\|_{L^2_{\text{mat}}} \leq C\|L_N\|_{L^2_{\text{mat}}\to L^2_{\text{mat}}} = O\left(e^{-cN}\right),$$

which implies $R = R_- = I + O(e^{-cN})$.

In order to prove the claim concerning $\partial_z R(z)$, we use the Cauchy integral formula

$$f'(z) = \oint_{\gamma} \frac{f(\omega)}{(\omega - z)^2}\frac{\mathrm{d}\omega}{2\pi\mathbb{i}}.$$

Let $\delta > 0$ be small enough such that $B(z,\delta)$ is entirely between the two circles in Γ, i.e. we have $1 - r < |z| \pm \delta < 1 + r$. According to Cauchy's integral formula and the result we proved above, we have

$$\begin{aligned}\partial_z R(z) &= \oint_{\partial B(z,\delta)} \frac{R(\omega)}{(\omega - z)^2} \frac{\mathrm{d}\omega}{2\pi \mathbb{i}} = \oint_{\partial B(z,\delta)} \frac{I + O(e^{-cN})}{(\omega - z)^2} \frac{\mathrm{d}\omega}{2\pi \mathbb{i}} \\ &= \oint_{\partial B(z,\delta)} \frac{I}{(\omega - z)^2} \frac{\mathrm{d}\omega}{2\pi \mathbb{i}} + \oint_{\partial B(z,\delta)} \frac{O(e^{-cN})}{(\omega - z)^2} \frac{\mathrm{d}\omega}{2\pi \mathbb{i}}.\end{aligned}$$

The first integral equals 0, as it is the derivative of a constant. The last integral equals $O(e^{-cN})$ by the uniformity of the O-term in z. $\square$

6.4 Second Szegő Theorem

Now we have all the pieces necessary to analyze the integrals

$$\oint_{|z|=1-\varepsilon} \left[Y(z;t)^{-1} \partial_z Y(z;t)\right]_{11} V(z) \frac{\mathrm{d}z}{2\pi \mathbb{i}} \tag{6.14}$$

and

$$\oint_{|z|=1+\varepsilon} \left[Y(z;t)^{-1} \partial_z Y(z;t)\right]_{11} V(z) \frac{\mathrm{d}z}{2\pi \mathbb{i}} \tag{6.15}$$

using T, S, R, and P. As we showed earlier, subtraction of these two integrals gives us what we need to find the product of the leading coefficients in the orthogonal polynomial related to the diagonal spin correlation.

Now we are ready to prove second Szegő theorem, also known as Szegő strong limit theorem. The result was first proven in [19].

Theorem 6.1 (Second Szegő Theorem). *Let χ_j be the leading coefficient of the orthogonal polynomial of degree j on the unit circle, with respect to weight function $w(e^{\mathbb{i}t}) = e^{V(e^{\mathbb{i}t})}$, that satisfies the conditions for the existence of orthogonal polynomials and has an analytic continuation in an annulus surrounding $\mathbb{T}$. Then, we have*

$$\log \prod_{j=0}^{N-1} \chi_j^{-2} = N\widehat{V}(0) + \sum_{k=1}^{\infty} k\widehat{V}(-k)\widehat{V}(k) + O\left(e^{-cN}\right).$$

Proof. Let us consider the case with $|z| = 1 - \varepsilon$ first. We write

$$Y(z;t) = S(z;t) = R(z;t)P(z;t)$$

and

$$\begin{aligned}Y(z;t)^{-1}\partial_z Y(z;t) &= P(z;t)^{-1}R(z;t)^{-1}\left(\partial_z R(z;t)P(z;t)+R(z;t)\partial_z P(z;t)\right)\\ &= O\left(e^{-cN}\right)+P(z;t)^{-1}\partial_z P(z;t),\end{aligned}$$

where Lemma 6.16 has been used to obtain the last equality. By applying this, we can write

$$\begin{aligned}&\oint_{|z|=1-\varepsilon}\left[Y(z;t)^{-1}\partial_z Y(z;t)\right]_{11}V(z)\frac{\mathrm{d}z}{2\pi\mathbb{i}}\\ &=\oint_{|z|=1-\varepsilon}\left[P(z;t)^{-1}\partial_z P(z;t)\right]_{11}V(z)\frac{\mathrm{d}z}{2\pi\mathbb{i}}+O\left(e^{-cN}\right).\end{aligned}$$

Note that the constant hidden in the definition of $O(e^{-cN})$ does not depend on N, t or z. Also, P is bounded.

Let us recall the expression for the matrix $P(z;t)$ from Lemma 6.12. Using that, we can write

$$P(z;t)^{-1}=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}e^{-t\oint_{\mathbb{T}}\frac{V(\omega)}{\omega-z}\frac{\mathrm{d}\omega}{2\pi\mathbb{i}}}&0\\0&e^{t\oint_{\mathbb{T}}\frac{V(\omega)}{\omega-z}\frac{\mathrm{d}\omega}{2\pi\mathbb{i}}}\end{pmatrix}.$$

Also, we have

$$\partial_z P(z;t)=\begin{pmatrix}t\partial_z\oint_{\mathbb{T}}\frac{V(\omega)}{\omega-z}\frac{\mathrm{d}\omega}{2\pi\mathbb{i}}e^{t\oint_{\mathbb{T}}\frac{V(\omega)}{\omega-z}\frac{\mathrm{d}\omega}{2\pi\mathbb{i}}}&0\\0&-t\partial_z\oint_{\mathbb{T}}\frac{V(\omega)}{\omega-z}\frac{\mathrm{d}\omega}{2\pi\mathbb{i}}e^{-t\oint_{\mathbb{T}}\frac{V(\omega)}{\omega-z}\frac{\mathrm{d}\omega}{2\pi\mathbb{i}}}\end{pmatrix}\begin{pmatrix}0&1\\-1&0\end{pmatrix}.$$

We combine the last two formulas and obtain

$$\begin{aligned}P(z;t)^{-1}\partial_z P(z;t) &= t\partial_z\oint_{\mathbb{T}}\frac{V(\omega)}{\omega-z}\frac{\mathrm{d}\omega}{2\pi\mathbb{i}}\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&1\\-1&0\end{pmatrix}\\ &= t\partial_z\oint_{\mathbb{T}}\frac{V(\omega)}{\omega-z}\frac{\mathrm{d}\omega}{2\pi\mathbb{i}}\begin{pmatrix}-1&0\\0&1\end{pmatrix},\end{aligned}$$

so we can see that

$$\left[Y(z;t)^{-1}\partial_z Y(z;t)\right]_{11}=-t\partial_z\oint_{\mathbb{T}}\frac{V(\omega)}{\omega-z}\frac{\mathrm{d}\omega}{2\pi\mathbb{i}}+O\left(e^{-cN}\right).$$

Now, we can write

$$\begin{aligned}&\int_0^1\oint_{|z|=1-\varepsilon}\left[Y(z;t)^{-1}\partial_z Y(z;t)\right]_{11}V(z)\frac{\mathrm{d}z}{2\pi\mathbb{i}}\mathrm{d}t\\ &=-\int_0^1 t\mathrm{d}t\oint_{|z|=1-\varepsilon}\left(\partial_z\int_{\mathbb{T}}\frac{V(\omega)}{\omega-z}\frac{\mathrm{d}\omega}{2\pi\mathbb{i}}+O\left(e^{-cN}\right)\right)V(z)\frac{\mathrm{d}z}{2\pi\mathbb{i}}\\ &=-\frac{1}{2}\oint_{|z|=1-\varepsilon}\partial_z\int_{\mathbb{T}}\frac{V(\omega)}{\omega-z}\frac{\mathrm{d}\omega}{2\pi\mathbb{i}}V(z)\frac{\mathrm{d}z}{2\pi\mathbb{i}}+O\left(e^{-cN}\right).\end{aligned}\tag{6.16}$$

Let us recall that we can express the function V as a Fourier series $V(\omega) = \sum_{k=-\infty}^{\infty} \widehat{V}(k)\omega^k$. Thus, we can see that

$$\oint_{\mathbb{T}} \frac{V(\omega)}{\omega - z} \frac{\mathrm{d}\omega}{2\pi\mathbb{i}} = J_{V,+}(z) = \sum_{k=0}^{\infty} z^k \widehat{V}(k),$$

which can be applied to (6.16) to obtain

$$\begin{aligned}
&-\frac{1}{2} \oint_{|z|=1-\varepsilon} \partial_z \left(\sum_{k=0}^{\infty} z^k \widehat{V}(k) \right) \left(\sum_{\ell=-\infty}^{\infty} z^\ell \widehat{V}(\ell) \right) \frac{\mathrm{d}z}{2\pi\mathbb{i}} \\
&= -\frac{1}{2} \oint_{|z|=1-\varepsilon} \left(\sum_{k=1}^{\infty} k z^k \widehat{V}(k) \right) \left(\sum_{\ell=-\infty}^{\infty} z^\ell \widehat{V}(\ell) \right) \frac{\mathrm{d}z}{2\pi\mathbb{i}z},
\end{aligned}$$

where the differentiation of a series term by term was allowed because the series converges for all $z \in \mathbb{C}$, which is since $|\widehat{V}(k)|$ decays fast enough by Lemma 5.3. Using the orthogonality of the terms of the two series and the dominated convergence theorem, this simplifies as

$$-\frac{1}{2} \sum_{k=1}^{\infty} k \widehat{V}(k) \widehat{V}(-k) \int_{|z|=1-\varepsilon} \frac{\mathrm{d}z}{2\pi\mathbb{i}z} = -\frac{1}{2} \sum_{k=1}^{\infty} k \widehat{V}(k) \widehat{V}(-k), \tag{6.17}$$

where the use of the dominated convergence theorem was acceptable due to Lemma 5.3 causing $k\widehat{V}(k)\widehat{V}(-k)$ to decrease faster than any polynomial.

Next, we perform similar computations for the case $|z| = 1 + \varepsilon$, i.e. the integral (6.15). This time we have

$$Y(z;t) = R(z;t)P(z;t) \begin{pmatrix} z^N & 0 \\ 0 & z^{-N} \end{pmatrix},$$

$$Y(z;t)^{-1} = \begin{pmatrix} z^{-N} & 0 \\ 0 & z^N \end{pmatrix} P(z;t)^{-1} R(z;t)^{-1},$$

and

$$\begin{aligned}
\partial_z Y(z;t) = \partial_z R(z;t) P(z;t) \begin{pmatrix} z^N & 0 \\ 0 & z^{-N} \end{pmatrix} + R(z;t) \partial_z P(z;t) \begin{pmatrix} z^N & 0 \\ 0 & z^{-N} \end{pmatrix} \\
+ N R(z;t) P(z;t) \begin{pmatrix} z^{N-1} & 0 \\ 0 & -z^{-N-1} \end{pmatrix}.
\end{aligned}$$

Multiplying these and applying Lemma 6.16 yields

$$\begin{aligned}\left[Y(z;t)^{-1}\partial_z Y(z;t)\right]_{11} =& O\left(e^{-cN}\right) + \left[\begin{pmatrix} z^{-N} & 0 \\ 0 & z^{N}\end{pmatrix} P(z;t)^{-1}\partial_z P(z;t) \begin{pmatrix} z^{-N} & 0 \\ 0 & z^{N}\end{pmatrix}\right]_{11} \\ &+ N\left[\begin{pmatrix} z^{-N} & 0 \\ 0 & z^{N}\end{pmatrix}\begin{pmatrix} z^{N-1} & 0 \\ 0 & -z^{-N-1}\end{pmatrix}\right]_{11} \\ =& O\left(e^{-cN}\right) + \left[P(z;t)^{-1}\partial_z P(z;t)\right]_{11} + \frac{N}{z}.\end{aligned}$$

Since z is outside the unit circle, the inverse and the z-derivative for $P(z;t)$ are

$$P(z;t)^{-1} = \begin{pmatrix} e^{-t\oint_{\mathbb{T}}\frac{V(\omega)}{\omega-z}\frac{\mathrm{d}\omega}{2\pi\mathbb{i}}} & 0 \\ 0 & e^{t\oint_{\mathbb{T}}\frac{V(\omega)}{\omega-z}\frac{\mathrm{d}\omega}{2\pi\mathbb{i}}}\end{pmatrix},$$

$$\partial_z P(z;t) = \begin{pmatrix} t\partial_z\oint_{\mathbb{T}}\frac{V(\omega)}{\omega-z}\frac{\mathrm{d}\omega}{2\pi\mathbb{i}}e^{t\oint_{\mathbb{T}}\frac{V(\omega)}{\omega-z}\frac{\mathrm{d}\omega}{2\pi\mathbb{i}}} & 0 \\ 0 & -t\partial_z\oint_{\mathbb{T}}\frac{V(\omega)}{\omega-z}\frac{\mathrm{d}\omega}{2\pi\mathbb{i}}e^{-t\oint_{\mathbb{T}}\frac{V(\omega)}{\omega-z}\frac{\mathrm{d}\omega}{2\pi\mathbb{i}}}\end{pmatrix}.$$

We multiply these two matrices together, which yields

$$\left[P(z;t)^{-1}\partial_z P(z;t)\right]_{11} = t\partial_z\oint_{\mathbb{T}}\frac{V(\omega)}{\omega-z}\frac{\mathrm{d}\omega}{2\pi\mathbb{i}},$$

and hence we have

$$\left[Y(z;t)^{-1}\partial_z Y(z;t)\right]_{11} = O\left(e^{-cN}\right) + t\partial_z\oint_{\mathbb{T}}\frac{V(\omega)}{\omega-z}\frac{\mathrm{d}\omega}{2\pi\mathbb{i}} + \frac{N}{z}.$$

We assume $|z|<1$, so according to (6.2), we have

$$\oint_{\mathbb{T}}\frac{V(\omega)}{\omega-z}\frac{\mathrm{d}\omega}{2\pi\mathbb{i}} = J_{V,-}(z) = -\sum_{k=-\infty}^{-1} z^k\widehat{V}(k).$$

At this point we are prepared to evaluate the integral

$$\begin{aligned}&\int_0^1\oint_{|z|=1+\varepsilon}\left[Y(z;t)^{-1}\partial_z Y(z;t)\right]_{11} V(z)\frac{\mathrm{d}z}{2\pi\mathbb{i}}\mathrm{d}t \\ =& \int_0^1 t\mathrm{d}t\oint_{|z|=1+\varepsilon}\partial_z\left(-\sum_{k=-\infty}^{-1} z^k\widehat{V}(k)\right)\left(\sum_{\ell=-\infty}^{\infty} z^\ell\widehat{V}(\ell)\right)\frac{\mathrm{d}z}{2\pi\mathbb{i}} + O\left(e^{-cN}\right) \\ &-\int_0^1\mathrm{d}t\oint_{|z|=1+\varepsilon}\frac{N}{z}V(z)\frac{\mathrm{d}z}{2\pi\mathbb{i}} + O\left(e^{-cN}\right).\end{aligned}$$

Using the orthonormality of the set of monomials z^k, we can simplify this as

$$\begin{aligned}
&\int_0^1 \oint_{|z|=1+\varepsilon} \left[Y(z;t)^{-1}\partial_z Y(z;t)\right]_{11} V(z)\frac{\mathrm{d}z}{2\pi\mathbb{i}}\mathrm{d}t \\
=&\frac{1}{2}\sum_{k=1}^{\infty} k\widehat{V}(-k)\widehat{V}(k) + N\widehat{V}(0) + O\left(e^{-cN}\right).
\end{aligned} \tag{6.18}$$

Combining the results (6.17) and (6.18) allow us to conclude that we have

$$\begin{aligned}
&\int_0^1 \left(\oint_{|z|=1+\varepsilon} \left[Y(z;t)^{-1}\partial_z Y(z;t)\right]_{11} V(z)\frac{\mathrm{d}z}{2\pi\mathbb{i}}\right. \\
&\left. - \oint_{|z|=1-\varepsilon} \left[Y(z;t)^{-1}\partial_z Y(z;t)\right]_{11} V(z)\frac{\mathrm{d}z}{2\pi\mathbb{i}}\right)\mathrm{d}t \\
=&N\widehat{V}(0) + \sum_{k=1}^{\infty} k\widehat{V}(-k)\widehat{V}(k) + O\left(e^{-cN}\right).
\end{aligned}$$

Plugging this into Lemma 6.9 completes the proof. □

Chapter 7

The Spin Correlation Function at a Sub-critical Temperature

At this point we have the necessary tools to continue analyzing the spin correlations at a subcritical temperature.

As a remark concerning the calculations below, the branch of the complex square root function has been chosen such that the branch cut is on $(-\infty, 0]$ and $\sqrt{x} > 0$ for all $x > 0$.

The following lemma provides an alternative representation for the weight function w. The parameters m and q are defined as in Theorem 3.2.

Lemma 7.1. *Let* $w\left(e^{\mathrm{i}t}\right) = (1+q^2)\left(1-\left(m\cos\frac{t}{2}\right)^2\right)^{\frac{1}{2}}$, *where* $m = 2(q+q^{-1})^{-1}$ *and* $0 < q < 1$. *It can be expressed as* $w\left(e^{\mathrm{i}t}\right) = \sqrt{1-q^2e^{\mathrm{i}t}}\sqrt{1-q^2e^{-\mathrm{i}t}}$.

Proof. By a direct computation, we have

$$
\begin{aligned}
&\sqrt{1-q^2e^{\mathrm{i}t}}\sqrt{1-q^2e^{-\mathrm{i}t}} = \sqrt{1+q^4-q^2e^{\mathrm{i}t}-q^2e^{-\mathrm{i}t}} \\
=&(1+q^2)\sqrt{\frac{1+q^4-q^2e^{\mathrm{i}t}-q^2e^{-\mathrm{i}t}}{(1+q^2)^2}} = (1+q^2)\sqrt{1-\frac{q^2(e^{\mathrm{i}t}+e^{-\mathrm{i}t}+2)}{1+2q^2+q^4}} \\
=&(1+q^2)\sqrt{1-\frac{e^{\mathrm{i}t}+e^{-\mathrm{i}t}+2}{\frac{1}{q^2}+2+q^2}} = (1+q^2)\sqrt{1-\frac{(e^{\frac{\mathrm{i}t}{2}}+e^{-\frac{\mathrm{i}t}{2}})^2}{\left(q+\frac{1}{q}\right)^2}} \\
=&(1+q^2)\sqrt{1-\left(m\cos\frac{t}{2}\right)^2} = w\left(e^{\mathrm{i}t}\right).
\end{aligned}
$$

□

Let us determine the Fourier series for the logarithmic weight function V next. Using the expression for $w(e^{\mathbb{i}t})$ from the previous lemma and the Taylor series for the logarithm, we can write

$$V\left(e^{\mathbb{i}t}\right) = \log w\left(e^{\mathbb{i}t}\right) = \frac{1}{2}\log\left(1 - q^2 e^{\mathbb{i}t}\right) + \frac{1}{2}\log\left(1 - q^2 e^{-\mathbb{i}t}\right)$$
$$= -\frac{1}{2}\sum_{k=1}^{\infty}\frac{q^{2k}}{k}e^{\mathbb{i}kt} - \frac{1}{2}\sum_{k=1}^{\infty}\frac{q^{2k}}{k}e^{-\mathbb{i}kt}. \tag{7.1}$$

From this Fourier series we can directly see that the Fourier coefficients are

$$\widehat{V}(k) = \widehat{V}(-k) = \begin{cases} -\frac{1}{2}\frac{q^{2k}}{k}, & k \geq 1 \\ 0, & k = 0. \end{cases}$$

Next, we define an auxiliary function that will play an important role in determining the spin correlation.

Definition 7.1. *Let us define a function $D : \mathbb{C} \setminus \mathbb{T} \to \mathbb{C}$ as*

$$D(z) = e^{\oint_{\mathbb{T}} \frac{\log w(\omega)}{\omega - z}\frac{d\omega}{2\pi\mathbb{i}}}.$$

The following lemma shows that the function D has properties that connect it to the weight function w. It also shows how a certain area integral can be expressed with the help of the Fourier coefficients of V. This alternative presentation appears in [3]. However, we will just directly use the Fourier coefficients later in the computation of D_n^*.

Lemma 7.2. *The function D has the following properties:*

1. $D(z) = \begin{cases} \left(1 - q^2 z\right)^{\frac{1}{2}}, & |z| < 1 \\ \left(1 - q^2 z^{-1}\right)^{-\frac{1}{2}}, & |z| > 1 \end{cases}$

2. $\frac{1}{\pi}\int_{|z|<1}\left|\frac{D'(z)}{D(z)}\right|^2 \mathrm{d}^2 z = \sum_{k=1}^{\infty} k|\widehat{V}(k)|^2.$

Proof. We can see in (7.1), that all the positive Fourier frequencies of $\log w(e^{\mathbb{i}t})$ correspond to the expression $\sqrt{1 - q^2 z}$, and all the negative frequencies to $\sqrt{1 - q^2 z^{-1}}$. Also, we have $D(z) = e^{I_V(z)}$, so by (6.2), we have

$$D(z) = e^{\oint_{\mathbb{T}} \frac{\log w(\omega)}{\omega - z}\frac{d\omega}{2\pi\mathbb{i}}} = \begin{cases} e^{\sum_{k=0}^{\infty}\widehat{V}(k)z^k} = \left(1 - q^2 z\right)^{\frac{1}{2}} & \text{for } |z| < 1 \\ e^{-\sum_{k=-\infty}^{-1}\widehat{V}(k)z^k} = \left(1 - q^2 z^{-1}\right)^{-\frac{1}{2}} & \text{for } |z| > 1. \end{cases}$$

To prove the second claim, we write

$$\log D(z) = \oint_{\mathbb{T}} \frac{V(\omega)}{\omega - z} \frac{\mathrm{d}\omega}{2\pi \mathbb{i}} = \sum_{k=0}^{\infty} \widehat{V}(k) z^k$$

for any $|z| < 1$. Therefore, we have

$$\frac{D'(z)}{D(z)} = \sum_{k=1}^{\infty} k \widehat{V}(k) z^{k-1}.$$

With the help of this representation we can write

$$\frac{1}{\pi} \int_{|z|<1} \left| \frac{D'(z)}{D(z)} \right|^2 \mathrm{d}^2 z = \frac{1}{\pi} \int_{|z|<1} \sum_{k,l=1}^{\infty} kl \widehat{V}(k) \overline{\widehat{V}(l)} z^{k-1} \overline{z}^{l-1} \mathrm{d}^2 z.$$

Next we use the dominated convergence theorem. This is possible by the smoothness of V and Lemma 5.3. We obtain

$$\frac{1}{\pi} \sum_{k,l=1}^{\infty} kl \widehat{V}(k) \overline{\widehat{V}(l)} \int_{|z|<1} z^{k-1} \overline{z}^{l-1} \mathrm{d}^2 z \tag{7.2}$$

The integral can be computed by changing variables to the polar coordinates such that $z = \rho e^{\mathbb{i}\varphi}$, which yields

$$\int_{|z|<1} z^{k-1} \overline{z}^{l-1} \mathrm{d}^2 z = \int_0^{2\pi} \int_0^1 \rho^{k-1} e^{\mathbb{i}(k-1)\varphi} \rho^{l-1} e^{-\mathbb{i}(l-1)\varphi} \rho \mathrm{d}\rho \mathrm{d}\varphi$$
$$= \int_0^{2\pi} e^{\mathbb{i}(k-l)\varphi} \mathrm{d}\varphi \int_0^1 \rho^{k+l-1} \mathrm{d}\rho = \begin{cases} \frac{\pi}{k}, & k = l \\ 0, & k \neq l. \end{cases}$$

We plug this into (7.2) and obtain

$$\sum_{k=1}^{\infty} k |\widehat{V}(k)|^2.$$

□

Now we have the results that we need in order to compute the diagonal spin-spin expectation in the subcritical case. The exact value was first published in [23].

Theorem 7.1. *For the inverse temperatures $\beta^* > \beta_{crit}$, we have*

$$\lim_{n \to \infty} D^*_{n+1} = \left(1 - q^4\right)^{\frac{1}{4}}.$$

Proof. Proposition 5.4 stated

$$D_{n+1}^* \Phi_n^*(q^2) + q^2 D_{n+1} \Phi_n(q^2) = \prod_{j=0}^{n} \|\Phi_j\|^2.$$

We want to solve D_{n+1}^*, which is the diagonal spin-correlation in the Ising model with free boundary conditions.

Let us show that we have $\lim_{n\to\infty} D_{n+1} = 0$. Recall that D_{n+1}^* is the diagonal spin correlation with free boundary conditions at the temperature β^*. Also, D_{n+1} is the correlation with $+$-boundary conditions in the inverse temperature β. The two inverse temperatures are connected by the relation $e^{-2\beta^*} = \tanh\beta$. We have $\beta = \beta^*$ only at the critical temperature. In order to compute the subcritical case, we assume $\beta^* > \beta_{\text{crit}}$, which implies $\beta < \beta_{\text{crit}}$. The function $D_{n+1}(\beta)$ is increasing with respect to β, which can be justified using Griffith's correlation inequality for the Ising model [13, Proposition IX.1], [11]. Thus, we have

$$D_{n+1}(0) \leq D_{n+1}(\beta) \leq D_{n+1}(\beta_{\text{crit}})$$

for $0 < \beta < \beta_{\text{crit}}$. Theorem 4.1 implies $D_{n+1}(\beta_{\text{crit}}) \to 0$. Also, from the definition of the Ising model we see that setting $\beta = 0$ turns the probability measure into a uniform distribution on the configuration space, implying zero correlation. Thus it has to be $D_{n+1}(\beta) \to 0$ for all $\beta < \beta_{\text{crit}}$.

The term $\Phi_n(q^2)$ with $|q^2| < 1$ can be computed as

$$\Phi_n(q^2) = Y_{11}(q^2) = [R(q^2;1)P(q^2;1)]_{11} = O\left(e^{-cN}\right),$$

where c is some constant independent from n. Thus, we conclude that we have $q^2 D_{n+1}\Phi_n(q^2) \to 0$ as $n \to \infty$, and

$$\lim_{n\to\infty} D_{n+1}^* = \lim_{n\to\infty} \frac{\prod_{j=0}^{n} \|\Phi_j\|^2}{\Phi_n^*(q^2)}, \tag{7.3}$$

if $\Phi_n^*(q^2)$ converges to a non-zero value.

Using (5.1), we write $\Phi_n^*(q^2) = q^{2n}\overline{\Phi_n(\overline{q}^{-2})}$, we can compute the denominator of (7.3) using the large-n asymptotics of $[Y(q^{-2})]_{11}$. Let $z = \frac{1}{q^2} > 1$. We have

$$\begin{aligned}
[Y(z)]_{11} &= \left[R(z;1)P(z;1)\begin{pmatrix} z^n & 0 \\ 0 & z^{-n}\end{pmatrix}\right]_{11} \\
&= \left[\left(I + O\left(e^{-cN}\right)\right)\begin{pmatrix} D(z) & 0 \\ 0 & D(z)^{-1}\end{pmatrix}\begin{pmatrix} z^n & 0 \\ 0 & z^{-n}\end{pmatrix}\right]_{11} \\
&= z^n D(z) + O(e^{-cN}).
\end{aligned}$$

Since $z > 1$, we can write $D(z) = \sqrt{1 - q^2 z^{-1}}$. On the other hand, Lemma 7.2 implies $D(1/q^2) = D(q^2)$. Thus, we have

$$\begin{aligned} \Phi_n^*\left(q^2\right) &= q^{2n}\Phi_n\left(\tfrac{1}{q^2}\right) = \tfrac{1}{q^{2n}} q^{2n} D\left(\tfrac{1}{q^2}\right) + O\left(e^{-cN}\right) \\ &= D\left(q^2\right)^{-1} + O\left(e^{-cN}\right) = \left(1 - q^4\right)^{-\frac{1}{2}} + O\left(e^{-cN}\right). \end{aligned}$$

Next, we compute the limit of the numerator in (7.3). We use Theorem 6.1, and obtain

$$\prod_{j=0}^{n} \|\Phi_j\|^2 = \prod_{j=0}^{n} \chi_j^{-2} = e^{n\widehat{V}(0) + \sum_{k=1}^{\infty} k|\widehat{V}(k)|^2 + O\left(e^{-cN}\right)}.$$

We see from the Fourier series in (7.1) that we have $\widehat{V}(0) = 0$, and

$$\sum_{k=1}^{\infty} k\left|\widehat{V}(k)\right|^2 = \sum_{k=1}^{\infty} k\left(-\frac{1}{2}\cdot\frac{q^{2k}}{k}\right)^2 = \frac{1}{4}\sum_{k=1}^{\infty}\frac{q^{4k}}{k} = -\frac{1}{4}\log\left(1 - q^4\right).$$

Thus, the limit is

$$\lim_{n\to\infty}\prod_{j=0}^{n}\|\Phi_j\|^2 = \lim_{n\to\infty} e^{-\frac{1}{4}\log\left(1-q^4\right) + O\left(e^{-cN}\right)} = \left(1 - q^4\right)^{-\frac{1}{4}}.$$

Now, we can compute

$$\lim_{n\to\infty} D_{n+1}^* = \frac{\lim_{n\to\infty}\prod_{j=0}^{n}\|\Phi_j\|^2}{\lim_{n\to\infty}\Phi_n^*\left(q^2\right)} = \frac{\left(1 - q^4\right)^{-\frac{1}{4}}}{\left(1 - q^4\right)^{-\frac{1}{2}}} = \left(1 - q^4\right)^{\frac{1}{4}}.$$

$\square$

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